hep-th/0608165

BTZ Black Hole with Gravitational Chern-Simons: Thermodynamics and Statistical Entropy

Mu-In Park[1]
Center for Quantum Spacetime,
Sogang University, Seoul 121-742, Korea

ABSTRACT

Recently, the BTZ black hole in the presence of the gravitational Chern-Simons (GCS) term has been studied and it has been found that the usual thermodynamical quantities, like as the black hole mass, angular momentum, and black hole entropy, are modified. But, for large values of the GCS coupling, where the modification terms dominate the original terms, some exotic behaviors occur, like as the roles of the mass and angular momentum are interchanged and the black hole entropy depends more on the *inner*-horizon area than the outer one. A basic physical problem of this system is that the form of entropy does not guarantee the second law of thermodynamics, in contrast to the Bekenstein-Hawking (BH) entropy. Moreover, this entropy does *not* agree with the statistical entropy, in contrast to a good agreement for small values of the GCS coupling. Here I find that there is another entropy formula where the usual BH form dominates the inner-horizon term again, as in the small GCS coupling, such as the second law of thermodynamics can be guaranteed. But now, the characteristic angular velocity and temperature are identified as those of the *inner* horizon, rather than the usual outer horizon, in order to satisfy the first law of thermodynamics. The temperature can have a *negative* value due to an *upper* bound of the mass as in spin systems and the angular velocity has a *lower* bound. Then, it is found that the new black hole entropy also agrees with the statistical entropy based on the holographic anomalies for the *whole* range of the GCS coupling. This reproduces, in the limit of vanishing Einstein-Hilbert term, the recent result about the exotic BTZ black holes where their masses and angular momenta are completely interchanged and the black hole entropies depend only on the area of the *inner* horizon. I compare the result of the holographic approach with the classical- symmetry-algebra-based approach and I find exact agreements even with the higher-derivative term of GCS. This provides a non-trivial check of

[1]Electronic address: muinpark@yahoo.com

the AdS/CFT-correspondence in the presence of higher-derivative terms in the gravity action. As a byproduct, I clarify how the correct “$1/\hbar$” factor in the semiclassical black hole entropy can be reproduced from the appropriate recovering of $\hbar$, which is “hidden” in the usual anomaly computations. I comment also about the reason of the general validity of the Cardy formula even with the higher-derivative/curvature corrections, its higher-order corrections, subtleties of extremal and near-extremal black holes, probing inside the horizon, and classical (in)stability problems.

PACS Nos: 04.60.-m, 04.70.Dy, 11.15.-q, 11.25.Hf
Keywords: Gravitational Chern-Simons, BTZ black hole, Black hole thermodynamics, Virasoro algebra, Cardy formula, Statistical entropy, AdS/CFT
25 August 2006

I. Introduction

The gravitational Chern-Simons (GCS) term in the Einstein gravity, with a *vanishing* cosmological constant Λ, produces a propagating, massive, spin-2 mode although the separate actions do not [1, 2, 3]. (This system is known as the "Topologically Massive Gravity (TMG)" in the literatures.) So, a massive object in this theory has the GCS *dressing* whose size is governed by the inverse of the graviton's mass, which is proportional to the coupling constant of the GCS term.

Recently, the BTZ black hole system as a *trivial* solution of the GCS-corrected gravity in the three-dimensional anti-de Sitter space (AdS) with a cosmological constant $\Lambda = -1/l^2$ (I call this *GCS-corrected/dressed BTZ (GCS-BTZ)* black hole) has been studied in the context of the higher-derivative/curvature gravities [4, 5, 6, 7, 8, 9, 10]. And it has been found that the usual thermodynamical quantities of the BTZ black hole, like as the black hole's mass, angular momentum, and entropy are modified as

$$M = m + xj/l, \ J = j + xlm, \tag{1.1}$$

$$S = \frac{2\pi r_+}{4G\hbar} + x\frac{2\pi r_-}{4G\hbar}, \tag{1.2}$$

which shows some *mixings* between the original BTZ black hole's mass m and angular momentum j, and also some deviation, proportional to the *inner*-horizon's area, from the usual Bekenstein-Hawking (BH) form [11, 12] in the black hole entropy [6, 8, 9, 10]. Here, the parameter x is proportional to the GCS coupling constant. These modifications would be the results of the GCS dressing in the *AdS* space, which have been absent in the usual TMG with $\Lambda = 0$.

But, that does not change much about the physical contents of the usual BTZ black hole when the parameter x is not large enough, more exactly when it is smaller than a critical value of the coupling constant. In fact, there is a good agreement in the entropy (1.2) with the statistical entropy based on the conformal field theory (CFT) for the Virasoro algebras at the spatially infinite boundary [8, 9], as in the usual BTZ black hole systems [13, 14, 15].

However, for large values of the GCS coupling, where the modification terms dominate the original terms, some exotic behaviors occur, like as the roles of mass and angular momentum are interchanged and its black hole entropy depends more on the *inner*-horizon area than the outer one. Actually similar phenomena have been also known for some time in several other contexts [16, 17, 18, 19] where the masses and angular momenta are *completely* interchanged and the black hole entropies depend *only* on the areas of the inner horizon (I have called these kinds of black holes as the *exotic black holes* [20]), in completely contrast to the BH's entropy formula [11, 12]. This looks similar to the suggestion in Ref. [21].

But a basic *physical* problem of those approaches is that the second law of thermodynamics is not guaranteed with their entropy formulae in contrast to the BH form [11]; actually, without the guarantee of the second law, there is no justification of identifying the entropies, even though they satisfy the first law, with the inner-horizon areas [12]. Moreover, those entropies do *not* agree with the statistical entropies, in contrast to a good agreement for small values of the GCS coupling, though this has not been well known in the literatures.

In the usual system of black holes, the first law of thermodynamics uniquely determines (up to an arbitrary constant) the black hole entropy with a given Hawking temperature T_+ and chemical potential for the outer (event) horizon r_+. In this context, there is no choice for the entropy of the GCS-BTZ black hole, other than (1.2), which is problematic for large values of x.

But recently, I have found that it is not the case for the exotic black holes [20], which corresponds to the $|x| \to \infty$ limit, by showing that there is another re-arrangement of the first law such as the entropy has the usual BH form, which is proportional to the area of the outer horizon, but now the characteristic temperature and chemical potential are those of the inner horizon, in contrast to the previous approaches. The temperature can have a *negative* value due to an upper bound of the mass as in spin systems and the angular velocity has a *lower* bound. And I have found that the new entropy formula have a good agreement with the statistical entropy based the CFT at the spatial infinity.

In this paper I show that the new approach can be generalized to large but *finite* values of x also: By considering the characteristic angular velocity and temperature as those of the inner horizon, a new entropy formula is found from the first law of thermodynamics. This new entropy agrees well with the statistical entropy. But, for small values of x, the system behaves like as an ordinary BTZ black hole with the characteristic angular velocity and temperature as those of the outer horizon with the known entropy formula (1.2), which agrees with the statistical entropy as well. So, I find two different phases of the GCS-BTZ black hole, depending on its GCS coupling constant. For each phase, the second law, as well as the first law, of the thermodynamics is guaranteed and there are good agreements with the statistical entropies.

The plan of this paper is as follows.

In Sec. II, I consider the thermodynamics of the GCS-BTZ black hole and find the new entropy formula for the large GCS coupling $|\hat{\beta}| > 1$, as well as the usual entropy formula for the small coupling $|\hat{\beta}| < 1$, from a new re-organization of the first law of thermodynamics. The inner horizon's temperature T_-, which is "negative" valued, (for $\hat{\beta} > 1$) or $-T_-$ (for $\hat{\beta} < -1$), and angular velocity Ω_- are considered as the characteristic parameters of the system for $|\hat{\beta}| > 1$, as well as the usual outer horizon's ones T_+, Ω_+ for $|\hat{\beta}| < 1$. I study also the Smarr formula and find the same form as in the usual BTZ black holes without the GCS term.

In Sec. III, the statistical entropy based on the holographic anomalies is considered and I find perfect agreements with the thermodynamic entropies that have been studied in Sec. II, for the *whole* range of the GCS coupling. The new entropy formula, as well as the ordinary one, is strongly supported by the CFT approach which is robust in the context of the AdS/CFT correspondence. And as a byproduct, I clarify how the correct "$1/\hbar$" factor in the semiclassical black hole entropy can be reproduced from the appropriate recovering of $\hbar$, which is "hidden" in the usual anomaly computations.

In Sec. IV, the classical symmetry algebra approach, based on the Chern-Simons formulation of three-dimensional gravity, is considered for comparison with the holographic anomaly approach of Sec. III and I find *exact* agreements between them. This provides a non-trivial check of the AdS/CFT-correspondence in the presence of higher-derivative terms in the gravity action. I include some details about the classical Kac-Moody and Virasoro algebra since there are several aspects which should be clarified though there are some recent works already in this direction. In order to ensure that the exact *factor* matching even with the GCS term is a solid result, by carefully fixing the subtleties involving the normalization differences between the different bases and conventions in the literatures, I include some details about the computations and useful formulae in Appendix **A**.

In Sec. V, I conclude with several discussions which include the reason of the general validity of the Cardy formula even with higher-derivative/curvature corrections, its higher-order corrections, subtleties of extremal and near-extremal black holes, probing inside the horizon by the GCS term, and classical (in)stability problems.

In Appendix **B**, I briefly review on the derivation of the Cardy formula and its higher-order corrections for completeness.

I shall omit the speed of light c and the Boltzman's constant k_B in this paper for convenience, by adopting the units of $c \equiv 1,\ k_B \equiv 1$. But, I shall keep the Newton's constant G and the Planck's constant $\hbar$ in order to clearly distinguish the quantum (gravity) effects with the classical ones.

II. Thermodynamics of the GCS-BTZ black hole

A. The BTZ black hole in the GCS-corrected gravity

The (2+1)-dimensional gravity with the GCS term and a cosmological constant $\Lambda = -1/l^2$ is described by the action on a manifold $\mathcal{M}$ [1, 2, 3] [omitting some possible boundary terms]

$$I_g = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) + I_{GCS}, \tag{2.1}$$

where the GCS term is given by [2] [the Greek letters $(\mu, \nu, \alpha, \cdots)$ denote the space-time in-

[2]Note that the *dimensioless* coupling constant $\hat{\beta} = x$ is related to the one used in Refs. [1, 2, 3] as

dices and Latin $(a, b, c, \cdots)$ denote the internal Lorentz indices; I take the metric convention η_{ab}=diag$(-1,1,1)$ for the internal Lorentz indices and the indices are raised and lowered by the metric η_{ab}]

$$I_{GCS} = \frac{\hat{\beta} l}{64\pi G} \int_{\mathcal{M}} d^3x \; \epsilon^{\mu\nu\alpha} \left(R_{ab\mu\nu} \omega^{ab}{}_{\alpha} + \frac{2}{3} \; \omega^{b}{}_{c\mu} \omega^{c}{}_{a\nu} \omega^{a}{}_{b\alpha} \right). \tag{2.2}$$

Here, the spin-connection $1-form$ $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu, \; \omega_{ab\mu} = -\omega_{ba\mu}$ is determined by the torsion-free condition

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{2.3}$$

with the dreibeins 1-form $e^a = e^a{}_\mu dx^\mu$, and the curvature is then $R_{ab\mu\nu} = \partial_\mu \omega_{ab\nu} + \omega_a{}^c{}_\mu \omega_{cb\nu} - (\mu \leftrightarrow \nu)$. [I take the same definitions as in Ref. [8] for the curvature 2-form $R_{ab} = (1/2) R_{ab\mu\nu} \; dx^\mu \wedge dx^\nu$ and the spin-connection 1-form ω_{ab}. Some useful formulae are summarized in Appendix **A**.] Note that I_{GCS} is of third-derivative order, rather than the second as in the Einstein-Hilbert term; so this is the first higher-derivative correction in three-dimensional spacetimes.

The resulting equations of motion, by varying I_g of (2.1) with respect to the metric[3], are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{l^2} g^{\mu\nu} = \hat{\beta} l C^{\mu\nu}, \tag{2.4}$$

where the Cotton tensor $C^{\mu\nu}$ is defined by

$$C^{\mu\nu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\rho\sigma} \nabla_\rho (R^\nu{}_\sigma - \frac{1}{4} \delta^\nu{}_\sigma R), \tag{2.5}$$

which is traceless and covariantly conserved [1]. From the fact that the Einstein equation (2.4) gives a constant curvature scalar $R = -6/l^2$, the equation (2.4) can be further reduced to

$$\begin{aligned} R^{\mu\nu} &= \frac{2}{l^2} g^{\mu\nu} + \hat{\beta} l C^{\mu\nu} \\ &= \frac{2}{l^2} g^{\mu\nu} + \frac{\hat{\beta} l}{\sqrt{-g}} \epsilon^{\mu\rho\sigma} \nabla_\rho R^\nu{}_\sigma. \end{aligned} \tag{2.6}$$

It would be a non-trivial task to find the general black hole solutions for the third-derivative-order equations[4]. However, there is a trivial solution, e.g., the BTZ solution because it satisfies the equation (2.6) trivially with $C^{\mu\nu} = \epsilon^{\mu\rho\sigma} \nabla_\rho R^\nu{}_\sigma / \sqrt{-g} = 0$ [4]. This looks like a too-trivial

$\hat{\beta} = -1/(\mu l)$, in Ref. [9] as $\hat{\beta} = -\beta_S / l$, and in Ref. [8] as $\hat{\beta} = -32\pi G \beta_{KL} / l$.

[3] The variations of I_{GCS} depends only the metric, though it does not look clear at first sight, as $\delta I_{GCS} = (l\hat{\beta}/8\pi G) \int_{\mathcal{M}} d^3x \sqrt{-g} \; C^{\mu\nu} \delta g_{\mu\nu}$ [1, 8].

[4] Recently, a non-trivial two-parameter family of black hole solutions have been found [22], but it does not seem that its properties have been fully elucidated yet.

situation which does not have any higher-derivative effect of the GCS term. But actually this is not the case, as we will see, since there are some non-trivial shifts in the physical parameters of the black hole [6, 8, 9]; the BTZ solution is rich enough to show some genuine effect of the GCS term. So, I concentrate hereafter only the BTZ solution, which is given by the metric [23, 24]

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{2.7}$$

with

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^\phi = -\frac{r_+ r_-}{l r^2}. \tag{2.8}$$

Here, r_+ and r_- denote the outer and inner horizons, respectively.

In the absence of the GCS term, the mass and angular momentum of the black hole are given by

$$m = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad j = \frac{2r_+ r_-}{8Gl}, \tag{2.9}$$

respectively. Note that these parameters satisfy the usual mass/angular momentum inequality $m^2 \geq j^2/l^2$ in order that the horizon exists or the conical singularity is not naked; the equality holds for the extremal black hole, where the inner and outer horizons overlap.

But, in the presence of the GCS term, it has been found that these "bare" parameters are shifted as (1.1)[5], i.e.,

$$M = m + \hat{\beta} j/l, \quad J = j + \hat{\beta} l m, \tag{2.10}$$

respectively; these modifications would be the results of GCS term in AdS space. One remarkable result of these modifications is that the usual mass/angualr momentum inequality is not valid generally

$$M^2 - J^2/l^2 = (1 - \hat{\beta}^2)(m^2 - j^2/l^2) \tag{2.11}$$

but it depends on the values of the GCS coupling constant $\hat{\beta}$: For small values of coupling $|\hat{\beta}| < 1$, the usual inequality is preserved, i.e., $M^2 \geq J^2/l^2$; however, for the large values of coupling $|\hat{\beta}| > 1$, one has an *anomalous* inequality with an exchanged role of the mass and angular momentum as $J^2/l^2 \geq M^2$; and also at the critical value $|\hat{\beta}| = 1$, the modified mass and angular momentum are "always" saturated, i.e., $M^2 = J^2/l^2$, regardless of inequality of

[5] This has been computed in several different approaches, e.g., the super-angular momentum's in Ref. [22], the quasi-local method's in Ref. [6], the ADM's in Refs. [25, 26], the holography's in Refs. [8, 9]. But they all give the same result.

the bare parameters m and j.

B. The black hole thermodynamics

Since the solution (2.7) has the same form of the metric as the usual BTZ solution, it has the same form of the Hawking temperature and angular velocity of the outer (event) horizon r_+ as the BTZ also:

$$T_+ = \frac{\kappa}{2\pi}\Big|_{r_+} = \frac{\hbar(r_+^2 - r_-^2)}{2\pi l^2 r_+}, \quad \Omega_+ = -N^\phi\Big|_{r_+} = \frac{r_-}{l r_+} \tag{2.12}$$

with the surface gravity function $\kappa = \partial N^2/(2\partial r)$.

Now, by considering the first law of thermodynamics as

$$\delta M = \Omega_+ \delta J + T_+ \delta S \tag{2.13}$$

with T_+ and Ω_+ as the characteristic temperature and angular velocity of the system, one can easily determine the black hole entropy as

$$S = \frac{2\pi r_+}{4G\hbar} + \hat{\beta}\frac{2\pi r_-}{4G\hbar}. \tag{2.14}$$

There is no other choice for the entropy in this usual context [16, 17, 18, 19, 9, 10]. In fact, this has been computed also in rather formal contexts like as the Euclidean method of conical singularity [9] and Wald's formalism [10] but the same entropy has been obtained.

However, an inherent problem of all those approaches is that there is *no* general proof about the second law of thermodynamics when higher-derivative/curvature terms are included in general [27, 28, 29]. In our case of (2.14), there are two contributions: One is the usual BH term

$$S_{BH} = \frac{2\pi r_+}{4G\hbar}, \tag{2.15}$$

which guarantees the second law from Hawking's area theorem [11, 12], which saying the increase of the area of the outer horizon $\mathcal{A}_+ = 2\pi r_+$. Another term is proportional to the inner-horizon area $\mathcal{A}_- = 2\pi r_-$ and this comes from the GCS term. But in this second part, the second law wold be questionable since some of the basic assumptions for the Hawking's area theorem, i.e., cosmic censorship conjecture might not be valid for the inner horizon in general. Moreover, the usual instability of the inner horizon makes it difficult to apply the Raychaudhuri's equation to get the area theorem, even without worrying about other assumptions for the theorem; actually this seems to be the situation that really occurs in our GCS-BTZ black holes also [30, 31].

But, there is a novel situation where the total entropy (2.14) still satisfies the second law, though all its constituents do not. This is the case where the usual BH term dominates the exotic term proportional to $\mathcal{A}_-$: Since $r_+ \geq r_-$ is always satisfied, this condition is equivalent to $|\hat{\beta}| < 1$. Actually, this is the case where the usual mass/angular momentum inequality holds, as I have shown in the previous sub-section II.A, the system behaves as an ordinary BTZ black hole though there are some shifts in the mass, angular momentum, and entropy.

On the other hand, for large values of coupling $|\hat{\beta}| > 1$, where the exotic term dominates the BH term, the above argument does not guarantee the second law of thermodynamics generally. Then, without the guarantee of the second law of thermodynamics, there is no justification of identifying entropy (2.14) even though it satisfies the first law of thermodynamics (2.13) and its characteristic temperature and angular velocity has the usual identifications [12].

So, in order to avoid the problem for the large couplings, we need another form of the entropy which is dominated by a term *linearly* proportional to the outer horizon area $\mathcal{A}_+$, following the Bekenstein's general argument [12], which should be valid in our case also; but the first law would be satisfied with some other appropriate temperature and angular velocity.

Recently, I have studied the extreme limit $|\hat{\beta}| \to \infty$ of the system and found that it is actually the case; and it can be generalized to our case also. A crucial fact for the new formulation is by observing the following identities in the BTZ system

$$\delta m = \Omega_+ \delta j + \frac{(r_+^2 - r_-^2)}{2\pi l^2 r_+} \left(\frac{2\pi \delta r_+}{4G} \right) \tag{2.16}$$

$$= \Omega_- \delta j + \frac{(r_-^2 - r_+^2)}{2\pi l^2 r_-} \left(\frac{2\pi \delta r_-}{4G} \right), \tag{2.17}$$

where

$$\Omega_- = -N^\phi \Big|_{r_-} = \frac{r_+}{l r_-} \tag{2.18}$$

is the angular velocity for the inner horizon; these identities show a symmetry between r_+ and r_-, which would reflect the symmetry in the metric (2.7), (2.8) and the bare parameters (2.9).

Then, the first identity (2.16) produces the usual first law of thermodynamics with the Hawking temperature T_+, and angular velocity Ω_+ for the outer horizon, and BH entropy S_{BH}:

$$\delta m = \Omega_+ \delta j + T_+ \delta S_{BH}. \tag{2.19}$$

The second identity (2.17) is an interesting re-arrangement of the first identity by replacing r_+ with r_-; this would be remarkable since the first law does *not* uniquely determine (up to a constant) the black hole entropy as well as the characteristic temperature and angular velocity in contrast to usual belief; actually the second identity (2.17) implies that the system can be also considered as a black hole with the entropy

$$S_- = \frac{2\pi r_-}{4G\hbar} \tag{2.20}$$

proportional to the inner-horizon area $\mathcal{A}_-$, and the characteristic temperature

$$T_- = \frac{\kappa}{2\pi}\Big|_{r_-} = \frac{\hbar(r_-^2 - r_+^2)}{2\pi l^2 r_-}, \tag{2.21}$$

angular velocity Ω_- for the inner horizon:

$$\delta m = \Omega_- \delta j + T_- \delta S_-. \tag{2.22}$$

Now, let me consider, from (2.10),

$$\delta M - \Omega_- \delta J = \delta m - \Omega_- \delta j + \hat{\beta}(\delta j/l - \Omega_- \delta m) \tag{2.23}$$

instead of $\delta M - \Omega_+ \delta J$ in (2.13). Then, it is easy to see that the first two terms in the right hand side become $T_- \delta S_-$ by using the second identity (2.17) or from (2.22). And also, the final two terms in the bracket become $T_- \delta S_{BH}$ by using the first identity (2.16) and another identity

$$\Omega_- = \Omega_+^{-1} l^{-2}. \tag{2.24}$$

So, finally I find that (2.23) becomes a new re-arrangement of the first law as

$$\delta M = \Omega_- \delta J + T_- \delta S_{\rm new} \tag{2.25}$$

with the new black hole entropy

$$S_{\rm new} = \frac{2\pi r_-}{4G\hbar} + \hat{\beta}\frac{2\pi r_+}{4G\hbar}. \tag{2.26}$$

With the above new entropy formula, it is easy to see that the previous argument for the second law of thermodynamics of (2.14) in the small values of coupling $|\hat{\beta}| < 1$ can now be applied to that of (2.26) in the large values of coupling $\hat{\beta} > 1$. But, with this new formulation, we have a dramatic departure from the usual situations. First, the angular velocity has a lower bound $\Omega_- \geq 1/l$ due to the fact of $r_+ \geq r_-$; it is saturated by the extremal case $r_+ = r_-$ and divergent for the vanishing inner horizon. This implies that this system is always rotating as far as there is an event horizon r_+. Second, the temperature T_- and surface gravity κ_- have "negative" values.[6] The negative-valued temperature looks strange in the usual black hole context, but this is a well-established concept in the spin systems where some *upper bound* of the energy level exists [32]. Actually this is exactly the same situation as in our case due to the upper bound of mass in (2.11), and this provides a physical justification of introducing

[6]I have used the definition of κ as $\nabla^\nu(\chi^\mu \chi_\mu) = -\kappa \chi^\nu$ for the horizon Killing vector χ^μ in order to determine its *sign*, as well as its magnitude.

the negative temperature in our system also.[7] This would be probably the first example in the black hole systems where the negative temperature occur.

On the other hand, for the large but "negative" values of coupling $\hat{\beta} < -1$, the entropy formula (2.26) wold *not* guarantee the second law of thermodynamics *nor* the *positiveness* of the entropy: The entropy would "decrease" indefinitely, with the negative values, as the outer horizon r_+ be increased following the area theorem. But there is a simple way of resolution from the new form of the first law (2.25). It is to consider[8]

$$S_{\text{new}}' \equiv -S_{\text{new}} = -\frac{2\pi r_-}{4G\hbar} - \hat{\beta}\frac{2\pi r_+}{4G\hbar}, \tag{2.27}$$

$$T_-' \equiv -T_- = \frac{\hbar(r_+^2 - r_-^2)}{2\pi l^2 r_-}, \tag{2.28}$$

instead of S_{new}, T_- and actually this choice is *unique*: One might consider $S_{\text{new}}'' \equiv \frac{2\pi r_-}{4G\hbar} - \hat{\beta}\frac{2\pi r_+}{4G\hbar}$ but the first law (2.25) is *not* satisfied then. Here, the temperature, as well as the entropy, is *positive* definite and this is consistent with the usual energy bound $M \geq J/l$, though M (and J also) can have *negative* values and satisfies the anomalous inequality $J^2/l^2 \geq M^2$.

C. The Smarr formula and its universality

So far, I have found that there are two different phases of the GCS-BTZ black hole, depending on its GCS coupling. The physics is quite different in the two phases having different thermodynamic functions, T_+, Ω_+, S for $|\hat{\beta}| < 1$ and $T_-, \Omega_-, S_{\text{new}}$ for $|\hat{\beta}| > 1$. But, for each phase, the second law, as well as the first law of thermodynamics, is guaranteed.

On the other hand, it is known that the bare BTZ black hole satisfies the three-dimensional Smarr formula [33]

$$m = \frac{1}{2}T_+ S_{BH} + \Omega_+ j. \tag{2.29}$$

So, an interesting question would be whether this formula is deformed in the presence of the higher-derivative/curvature terms in the action; and also the study of this relation would be important in that it could show some universal characteristics of the system in connection with other thermodynamical systems which look completely different.

This would be a non-trivial question in the general asymptotically-AdS space [34, 35]. But, in our GCS-BTZ case, the Smarr formula (2.29) is unchanged from some magic of the system. The magic comes first from the following identity, in addition to (2.29),

$$m = \frac{1}{2}T_- S_- + \Omega_- j \tag{2.30}$$

[7] The positive-valued surface gravity and temperature with $T = |\kappa_-/(2\pi)|$ (as in Ref. [31]) produces an incorrect sign in front of the TdS term in (2.25).

[8] This has been inspired by the discussion with S. Odintsov. I thank him for the discussion.

and this can be considered as another re-arrangement of the three-dimensional Smarr formula (2.29), which has never been considered in the literatures.[9] And also, by considering (2.24) and $T_-\Omega_-^{-1} = -T_+/l$, one has the identities

$$\begin{aligned} j/l &= \frac{1}{2}T_-S_+ + l\Omega_- m && (2.31)\\ &= \frac{1}{2}T_+S_- + l\Omega_+ m. && (2.32)\end{aligned}$$

The first and second identities come from (2.29) and (2.30), respectively.

Then, from all these magical identities, one can easily find the following Smarr formulae for the GCS-corrected mass and angular momentum, M and J, respectively

$$\begin{aligned} M &= \frac{1}{2}T_+S + \Omega_+ J, && (2.33)\\ M &= \frac{1}{2}T_-S_{\text{new}} + \Omega_- J && (2.34)\\ &= \frac{1}{2}T_-{}'S_{\text{new}}{}' + \Omega_- J && (2.35)\end{aligned}$$

by considering (2.29), (2.32), and (2.30), (2.31), respectively. Here, (2.29) and (2.33) describe the black holes with $|\hat{\beta}| < 1$ since T_+ and Ω_+ are considered as the characteristic parameters of the system. Similarly, (2.30) and (2.34) describe those with $|\hat{\beta}| > 1$.

So, I have found that the two Smarr formulae (2.29) and (2.30) extend to the GCS-BTZ black hole with the corrected M, J and the entropies S or S_{new}, $S_{\text{new}}{}'$.[10] However, it is not clear whether the *covariance* of Smarr formula is just a result of the speciality of the GCS term or there are other deep reasons.

III. Statistical entropy: the holographic anomaly approach

In the usual context of the AdS/CFT correspondence [36], the central charges for the CFT on the asymptotic AdS boundary are identified by evaluating the anomalies of the CFT effective action, from the regularized bulk gravity action [37, 38, 39]. (See also Refs. [40, 41] for some alternatives approaches.)

Recently, the approach has been applied to the action (2.1) and it is found that there are anomalies in the expectation values of the boundary stress tensor, for the boundary metric

[9] This would correspond to (2.17) in the differential form as (2.29) do for (2.16) and so these "dual" descriptions would be closely related to those of the first law. But, I do not see any "theoretical" reason why the general variations of (2.29) and (2.30) give (2.16) and (2.17), respectively, beyond considering specific solutions.

[10] Of course, other forms of Smarr-like formula with mixed combinations might be considered mathematically, but this would not be interesting physically since there are more than two independent parameters in the Smarr-like formula.

$ds^2 \simeq -r^2 dx^+ dx^-$ with r taken to infinity,

$$\left\langle T_{++}(x^+)\right\rangle = -\frac{\hbar\hat{c}^+}{24\pi}, \quad \left\langle T_{--}(x^+)\right\rangle = -\frac{\hbar\hat{c}^-}{24\pi} \tag{3.1}$$

with the central charges [I follow the conventions of [39]]

$$\hat{c}^{\pm} = \gamma^{\pm}\frac{3l}{2G\hbar} \tag{3.2}$$

with $\gamma^{\pm} = 1 \pm \hat{\beta}$ for the right/left-moving sectors with the superscripts $+$ and $-$, respectively. Note that the obtained central charges have the *quantum* origin, which would has been introduced via the regularization procedure; the anomaly equations (3.1) have been written, usually, as if no $\hbar$ is involved, e.g., $\langle T_{\pm\pm}\rangle = -c^{\pm}/(24\pi)$, with classical numbers $c^{\pm}$, so it was quite unclear how to recover the Planck's constant $\hbar$ to reproduce the correct "$1/\hbar$" factor of the semiclassical black hole entropy, like as the BH entropy (2.15), via the Cardy formula.

By considering (3.1) as the anomalous transformations of the boundary stress tensors under the diffeomorphism $\delta x^{\pm} = -\xi^{\pm}(x^{\pm})$,

$$\begin{aligned}
\delta_{\xi^+} T_{++} &= 2\partial_+\xi^+ T_{++} + \xi^+\partial_+ T_{++} - \frac{\hbar\hat{c}^+}{24\pi}\partial_+^3\xi^+ \\
&= \frac{1}{i}[T_{++}, \hat{L}^+[\xi^+]], \\
\delta_{\xi^-} T_{--} &= 2\partial_-\xi^- T_{--} + \xi^-\partial_- T_{--} - \frac{\hbar\hat{c}^-}{24\pi}\partial_-^3\xi^- \\
&= \frac{1}{i}[T_{--}, \hat{L}^-[\xi^-]]
\end{aligned} \tag{3.3}$$

with the generators

$$\hat{L}^{\pm}[\xi^{\pm}] = \frac{1}{\hbar}\oint dx^{\pm} T_{\pm\pm}\xi^{\pm}(x^{\pm}) + \frac{\hat{c}^{\pm}}{24}, \tag{3.4}$$

one can obtain a pair of quantum Virasoro algebras

$$[\hat{L}^{\pm}_m, \hat{L}^{\pm}_n] = (m-n)\hat{L}^{\pm}_{m+n} + \frac{\hat{c}^{\pm}}{12} m(m^2-1)\delta_{m+n,0} \tag{3.5}$$

for a monochromatic basis $\xi^{\pm} = e^{imx^{\pm}}$ with the integer numbers m, n. Here I note that this reduces to the usual result for the holographic *conformal* anomaly in the $\hat{\beta} \to 0$ limit [37, 38, 39], whereas $\hat{\beta}$-dependent terms come from the holographic *gravitational* anomaly due to the GCS term [8, 9].

Now, let me consider the ground state Virasoro generators, expressed in terms of the black hole's mass and angular momentum:

$$\begin{aligned}
\hat{L}^{\pm}_0 &= \frac{lM \pm J}{2\hbar} + \frac{\hat{c}^{\pm}}{24} \\
&= \gamma^{\pm}\frac{(lm \pm j)}{2\hbar} + \frac{\hat{c}^{\pm}}{24}.
\end{aligned} \tag{3.6}$$

With the Virasoro algebras of $\hat{L}^{\pm}_m$ in the standard form, which are defined on the *plane*, one can use the Cardy formula for the asymptotic states [42, 43, 44, 45, 46]

$$\log\,\rho(\hat{\Delta}^{\pm}) \simeq 2\pi\sqrt{\frac{1}{6}\left(\hat{c}^{\pm}-24\hat{\Delta}^{\pm}_{\rm min}\right)\left(\hat{\Delta}^{\pm}-\frac{\hat{c}^{\pm}}{24}\right)}, \tag{3.7}$$

where $\hat{\Delta}^{\pm}$ are the eigenvalues, called conformal weights, of the operator $\hat{L}_0$ for black-hole quantum states $|\hat{\Delta}^{\pm}\rangle$ and $\hat{\Delta}^{\pm}_{\rm min}$ are their minimum values. Here, I note that the above Cardy formula, which comes from the saddle-point approximation of the CFT partition function on a torus, is valid only if the following two conditions are satisfied:

$$\frac{24\hat{\Delta}^{\pm}_{\rm eff}}{\hat{c}^{\pm}_{\rm eff}} \gg 1, \tag{3.8}$$

$$\hat{c}^{\pm}_{\rm eff}\hat{\Delta}^{\pm}_{\rm eff} \gg 1, \tag{3.9}$$

where $\hat{\Delta}^{\pm}_{\rm eff}=\hat{\Delta}^{\pm}-\hat{c}^{\pm}/24,\ \hat{c}^{\pm}_{\rm eff}=\hat{c}^{\pm}-24\hat{\Delta}^{\pm}_{\rm min}$ are the effective conformal weights and central charges, respectively; from the first condition, the higher-order correction terms are exponentially suppressed as $e^{-2\pi\epsilon^{\pm}(\hat{\Delta}^{\pm}-\hat{\Delta}^{\pm}_{\rm min})}$ with $\epsilon^{\pm}\equiv 24\hat{\Delta}^{\pm}_{\rm eff}/\hat{c}^{\pm}_{\rm eff}$; from the second condition, the usual saddle-point approximation is reliable, i.e., $\rho(\hat{\Delta}^{\pm})$ dominates in the partition function (see Appendix **B** for the details).

Then, the statistical entropy for the asymptotic states becomes

$$\begin{aligned} S_{\rm stat} &= \log\,\rho(\hat{\Delta}^{+}_0)+\log\,\rho(\hat{\Delta}^{-}_0) \\ &= \frac{\pi}{4G\hbar}|\gamma^{+}(r_{+}+r_{-})|+\frac{\pi}{4G\hbar}|\gamma^{-}(r_{+}-r_{-})| \\ &= \frac{\pi}{4G\hbar}(|\gamma^{+}|+|\gamma^{-}|)r_{+}+\frac{\pi}{4G\hbar}(|\gamma^{+}|-|\gamma^{-}|)r_{-}\ , \end{aligned} \tag{3.10}$$

where I have chosen $\hat{\Delta}^{\pm}_{0\ ({\rm min})}=0$ as usual [13, 14, 15]; from (3.6) this corresponds to the AdS_3 vacuum solution where $m=-1/(8G)$ and $j=0$ in the usual context, but it has a permanent rotation as well in the new context [8],

$$M=-\frac{1}{8G},\quad J=-\frac{l\hat{\beta}}{8G}. \tag{3.11}$$

Note that the correct "$1/\hbar$" factor for the semiclassical black hole entropy comes from the appropriate recovering of $\hbar$ in (3.2) and (3.6). According to the conditions of validity (3.8), (3.9), this entropy formula is valid only when both of the two conditions

$$(r_{+}\pm r_{-})\gg l, \tag{3.12}$$

$$(r_{+}\pm r_{-})\gg \hbar G \tag{3.13}$$

are satisfied. The usual semiclassical limit of large black hole (area), in which the back-reaction of the emitted radiation from the black hole is neglected [47] and so the thermodynamical entropy formula (2.14) and (2.26) from the first law can be reliable, agrees with the condition (3.13) and so there would be no obstacles to compare the statistical entropy (3.10) with the thermodynamical one. Note that from another condition (3.12) we are considering a more restricted class of black hole systems[11], though this does not seem to be needed in general.

Now, let me consider the following four cases, depending on the values of $\hat{\beta}$: (a). $|\hat{\beta}| < 1$, (b). $\hat{\beta} > 1$, (c). $\hat{\beta} < -1$, and (d). $|\hat{\beta}| = 1$.

(a). In this case, I have $|\gamma^{\pm}| = \gamma^{\pm}$ and the statistical entropy (3.10) becomes

$$S_{\rm stat} = \frac{2\pi r_+}{4G\hbar} + \hat{\beta}\frac{2\pi r_-}{4G\hbar} \tag{3.14}$$

from $\gamma^+ + \gamma^- = 2,\ \gamma^+ - \gamma^- = 2\hat{\beta}$. This agrees exactly with the usual entropy formula (2.14). And this is the case where $\hat{c}^{\pm}$ and $\hat{\Delta}^{\pm} - \hat{c}^{\pm}/24$ are positive definite such as the Cardy formula (3.7) has a well-defined meaning. In the gravity side also it shows the usual behavior with the "positive" mass and angular momentum satisfying the normal inequality $M^2 \geq J^2/l^2$.

(b). In this case, I have $|\gamma^+| = \gamma^+,\ |\gamma^-| = -\gamma^-$ and so the statistical entropy (3.10) becomes

$$S_{\rm stat} = \frac{2\pi r_-}{4G\hbar} + \hat{\beta}\frac{2\pi r_+}{4G\hbar}. \tag{3.15}$$

This agrees exactly with the new entropy formula (2.26), which guarantees the second law of thermodynamics even in this case. And this is the case where there is some abnormal change of the role of the mass and angular momentum due to $J^2/l^2 \geq M^2$ even though M and J both are positive definite as usual. Moreover, in the CFT side also, this is not the usual system because $\hat{c}^- = \gamma^- 3l/(2G\hbar)$ and $\hat{\Delta}^- - \hat{c}^-/24 = \gamma^-(ml - j)/2\hbar$ are *negative* valued though their self-compensations of the negative signs produce the *real* and *positive* statistical entropy. The application of the Cardy formula to the case of negative $\hat{c}^-$ and $\hat{\Delta}^- - \hat{c}^-/24$ might be questioned due to the existence of negatives-norm states with the usual condition $\hat{L}^-_n|\hat{\Delta}^-\big\rangle = 0$ $(n > 0)$ for the highest-weight state $|\hat{\Delta}^-\big\rangle$. However, this problem can be easily cured by considering another representation of the Virasoro algebra with $\tilde{L}^-_n \equiv -\hat{L}^-_{-n},\ \tilde{c}^- \equiv -\hat{c}^-$ and $\tilde{L}^-_n|\tilde{\Delta}^-\ \rangle = 0$ $(n > 0)$ for the new highest-weight state $|\tilde{\Delta}^-\ \rangle$ [48]; this implies that the Hilbert space need to be "twisted" in which the whole states vectors be constructed from the twisted highest-weight state $|\hat{\Delta}^+\big\rangle \otimes |\tilde{\Delta}^-\big\rangle$. The formula (3.10), which is invariant under this

[11] At this state, the condition of large central charges $\hat{c}^{\pm} \gg 1$, i.e., $l \gg \hbar G$ [13], which would be related to the leading supergravity approximation of AdS/CFT correspondence [36], is not needed yet.

substitution, should be understood in this context. On the other hand, if I take the limit $\hat{\beta} \to \infty$, in which there is only the GCS term, this becomes the "exotic" black hole system that occur in several different contexts [16, 17, 18, 19, 20]; but note that this can *not* be obtained from (3.14).

(c). In this case, I have $|\gamma^+| = -\gamma^+,\ |\gamma^-| = \gamma^-$ and the statistical entropy (3.10) becomes

$$S_{\text{stat}} = -\frac{2\pi r_-}{4G\hbar} - \hat{\beta}\frac{2\pi r_+}{4G\hbar}. \tag{3.16}$$

Note that this is positive definite and this should be the case from its definition $S_{\text{stat}} = \log(\rho(\hat{\Delta}_0^+)\rho(\hat{\Delta}_0^-)) \geq 0$ for the number of possible states $\rho(\hat{\Delta}_0^\pm) \geq 1$. This agrees exactly with the new entropy formula (2.27), which guarantees the second law. And this is the case where M *can* be negative and J has the opposite direction to the bare one j, in contrast to the positive definite M and J in the cases of (a) and (b), as well as the anomalous inequality $J^2/l^2 \geq M^2$. In the CFT side, $\hat{c}^+$ and $\hat{\Delta}^+ - \hat{c}^+/24$ become negative-valued now and I need to twist this right-moving sector, rather than the left-moving one as in the case (b), $\tilde{L}_n^+ \equiv -\hat{L}_{-n}^+,\ \tilde{c}^+ \equiv -\hat{c}^+$ and $\tilde{L}_n^+|\tilde{\Delta}^+\ \rangle = 0\ (n > 0)$ for the twisted highest-weight state $|\tilde{\Delta}^+\big\rangle \otimes |\hat{\Delta}^-\big\rangle$.

(d). In this case, one of $\gamma^\pm$ vanishes, i.e., $\gamma^+ = 0,\ \gamma^- = 2$ for $\hat{\beta} = -1$ and $\gamma^+ = 2,\ \gamma^- = 0$ for $\hat{\beta} = 1$. The statistical entropy becomes

$$S_{\text{stat}} = \frac{2\pi}{4G\hbar}(r_+ - r_-) \quad (\hat{\beta} = -1), \tag{3.17}$$

$$S_{\text{stat}} = \frac{2\pi}{4G\hbar}(r_+ + r_-) \quad (\hat{\beta} = +1). \tag{3.18}$$

Note that (3.18) can be reproduced from (3.14) and (3.15), but (3.17) from (3.14) and (3.16). So, statistical entropies (3.17) and (3.18) agree exactly with the usual entropy formula (3.14)$\sim$(3.16). As I have remarked previously in Sec. II. (b), this is the case where the mass/angular momentum inequality saturates $M^2 = J^2/l^2$ regardless of m and j. In fact, they satisfy

$$M = \pm J/l = \frac{(r_+ \pm r_-)^2}{8Gl^2} \geq 0 \tag{3.19}$$

for $\hat{\beta} = \pm 1$, respectively. So, for non-extremal bare black holes with $r_+ > r_-$, the mass M is positive definite but J changes its direction for $\hat{\beta} = -1$. For extremal bare black holes with $r_+ = r_-$, one has $M = J = 0$ as well as $S_{\text{stat}} = S = 0$ satisfying the Nernst formulation of the third law of thermodynamics [49, 50, 51] for $\hat{\beta} = -1$, whence $M = J = (G\hbar/(2\pi^2 l^2))S_{\text{stat}} = r_+^2/(2Gl^2) > 0$ without satisfying the third law for $\hat{\beta} = 1$ as in all other cases of (a)$\sim$(c) and in the usual Kerr black hole [52]. But, there are some subtleties about this in the fully corrected entropies; see Sec. V about this issue.

In summary, I have found exact agreements between the thermodynamic black hole entropies which have been evaluated in the bulk (AdS) gravity side and the CFT entropies in the asymptotic boundary, for the *whole* range of the coupling constant $\hat{\beta}$. So, the new entropy formula for the strong coupling $|\hat{\beta}| > 1$ with the unusual characteristic temperature and angular momentum (T_-, Ω_-) or $(T_-{}', \Omega_-)$ is strongly supported by the CFT approach also. This reveals the AdS/CFT correspondence in the sub-leading order with the higher-derivative term of GCS as well as in the leading order with the Einstein-Hilbert action.

IV. Comparison with the classical symmetry algebra approach: Exact agreements

There is an alternative approach to compute the Virasoro algebras with the central charges. This is based on the *classical* symmetry algebras of the asymptotic isometry of AdS_3 [53, 54, 55, 56, 57]

$$\{L_m^{\pm}, L_n^{\pm}\}^* \approx i(m-n)L_{m+n}^{\pm} + \frac{ic^{\pm}}{12} m(m^2-1)\delta_{m+n,0} \tag{4.1}$$

with the "classical" central charges $c^{\pm}$ and the Dirac bracket $\{\ ,\ \}^*$ [58].

It is well known that there is an exact agreement with the anomaly based approaches of Sec. III by the mapping [43], with the appropriate recovering of $\hbar$,

$$\hat{c}^{\pm} = \frac{c^{\pm}}{\hbar}, \quad \hat{L}_m^{\pm} = \frac{L^{\pm}}{\hbar} \tag{4.2}$$

in the absence of the GCS term [37, 38, 39] [12]. So, the statistical entropy agrees with the BH entropy also. But, this is a quite non-trivial fact and actually this provides an *explicit* check of the AdS/CFT correspondence by comparing the classical data $(c^{\pm}, L^{\pm})$, which can be *directly* computed, with the quantum data $(\hat{c}^{\pm}, \hat{L}^{\pm})$ in the anomaly approach, which can be identified *only indirectly* through the (conjectured) AdS/CFT-correspondence.

So, it would be interesting to consider the classical approach in the presence of the GCS term also, and compare with the results from the anomaly approach of Sec. III in order to see whether they both agree or not. This would provide a non-trivial check of the AdS/CFT-correspondence beyond the Einstein-Hilbert action; there are some works already in this direction [5, 64, 65] but there are several aspects which should be clarified.

[12] The classical algebra with the higher curvature terms was computed in Ref. [59] by transforming the gravity action with the higher curvature terms into the usual Einstein-Hilbert action with some auxiliary tensor matter fields. The same central charges and Virasoro generators have been obtained in the anomaly approach also recently [7]. But the validity of Ref. [59] is unclear since there would be non-trivial contributions in the generators $L_m^{\pm}$ and central charges from the matter fields *in general* though the agreement seems to be plausible in the context of AdS/CFT [15, 60, 61, 62, 63].

There are two "classically" equivalent approaches for this purpose: They are the purely gravity approach of Brown-Henneaux [53] and Chern-Simons (CS) approach. I consider the latter approach here since it is easier and provides some explicit computations of the symmetry generators and their Dirac brackets of (4.1) even *far* from asymptotic boundary, which are *not* available in the former approach. Moreover, it can reveal the holographic phenomena explicitly and the novel boundary effects to the derivative of Dirac delta function, which are the mathematical origin of the classical central terms [56].

A. The Chern-Simons gauge theory with boundaries

It is well known that the CS (gauge) theory with boundaries produces the central terms in the Virasoro algebras, as well as in the Kac-Moody algebras even at the "classical" level; this has been first spelled out in [54] but rigorously computed later in [56, 57]. This is a general field theoretic result only if some appropriate boundary conditions are satisfied regardless of the physical contents of the CS theory. [This boundary effect should be distinguished from the "finite size effect"; it exits even for an infinitely large boundary.] Moreover, this is *not* an artifact of a "classical" theory but persists even in the quantum theory because it can not be removed from some quantum effects due to normal orderings [57].

So, if a theory can be expressed as the CS theory with the appropriate boundary conditions, one can quickly identify the Kac-Moody and Virasoro algebras with the classical central terms. This is actually the case of three-dimensional Einstein gravity with the cosmological constant Λ, where the usual BTZ black hole or the three-dimensional Kerr-de Sitter solutions (KdS_3) are admitted, depending on the sign of Λ [54, 56].

The generalization of this approach to some more general class of gravity systems, i.e., with the matter couplings [15] or with the higher curvature terms [59, 7, 10] would not be possible in general. But, the three-dimensional gravity with the GCS term is an exceptional case since the GCS term itself can also be expressed as the CS theory for another choice of the invariant quadratic forms of the Lie algebra for non-vanishing Λ [66]; for $\Lambda = 0$, the quadratic forms are not well defined since they are degenerate. So, for the most general form of the invariant quadratic forms which admit the new choice for the GCS action as well, one can express the Einstein gravity with the GCS term and non-vanishing Λ as a CS gauge theory [66, 5].

Moreover, in the GCS-BTZ black holes, there is no difference in the metric form though there are some shifts in the ADM mass and angular momentum and so there is no difference in the boundary conditions for the corresponding CS theory; but this would not be valid generally for other non-trivial solutions where there are some important deformations of the metric itself.

Hence, all the previous results about the bare BTZ black hole can be applied to the GCS-BTZ case also from the general results of the Kac-Moody and Virasoro algebras for the CS theory.

B. The SO(2,2) Chern-Simons gravity with the GCS term

For the (2+1)-dimensional space with a negative cosmological constant $\Lambda = -1/l^2$, the symmetry of the space is $SO(2,2)$ [$SO(3,1)$ for a positive cosmological constant], which has the following commutation relations among the generators of the Lie group

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \quad [J_a, P_b] = \epsilon_{ab}{}^c P_c, \quad [P_a, P_b] = \frac{1}{l^2}\epsilon_{ab}{}^c J_c. \tag{4.3}$$

The most general form of the invariant quadratic forms are [66, 5]

$$\langle J_a, J_b\rangle = \alpha\eta_{ab}, \quad \langle J_a, P_b\rangle = \beta\eta_{ab}, \quad \langle P_a, P_b\rangle = \frac{\alpha}{l^2}\eta_{ab}. \tag{4.4}$$

Here α and β are some arbitrary constants but the ratio of $\langle J_a, J_b\rangle$ and $\langle P_a, P_b\rangle$ are completely fixed by the algebras (4.3).

The algebras (4.3) and the quadratic forms (4.4) look unusual. But if I introduce

$$J_a^{\pm} = \frac{1}{2}(J_a \pm l P_a), \tag{4.5}$$

(4.3) and (4.4) become

$$[J_a^{\pm}, J_b^{\pm}] = \epsilon_{ab}{}^c J_c^{\pm}, \quad [J_a^{\pm}, J_b^{\mp}] = 0, \tag{4.6}$$

$$\left\langle J_a^{\pm}, J_b^{\pm}\right\rangle = \frac{1}{2}(\alpha \pm \beta l)\eta_{ab}, \quad \left\langle J_a^{\pm}, J_b^{\mp}\right\rangle = 0. \tag{4.7}$$

This is the usual form of the $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ Lie algebra but with different values of the quadratic forms of the two sectors.

Now, by considering the Lie-algebra-valued one-form

$$\begin{aligned} \mathbf{A} &= \omega^a J_a + e^a P_a = A^+ + A^-, \\ A^{\pm} &= (\omega^a \pm \frac{e^a}{l}) J_a^{\pm} \end{aligned} \tag{4.8}$$

with the triads $e^a = e^a{}_\mu dx^\mu$ and the spin connections $\omega^a = (1/2)\epsilon^{abc}\omega_{\mu bc}dx^\mu$ [13], the CS action becomes [$\langle A \wedge B\rangle$ is understood as $\langle A \wedge, B\rangle$], up to some boundary terms,

$$\begin{aligned} I_{CS}[\mathbf{A}] &= \frac{k}{4\pi}\int_{\mathcal{M}} \left\langle \mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\,\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right\rangle \\ &= \frac{k}{4\pi}\Omega^+ \int_{\mathcal{M}} Tr\left(A^+ \wedge dA^+ + \frac{2}{3}\,A^+\wedge A^+\wedge A^+\right) - \quad (+\leftrightarrow -) \\ &= \frac{k\beta}{\pi}\int_{\mathcal{M}} Tr\left(e\wedge R + \frac{1}{3}\,e\wedge e\wedge e\right) \\ &\quad + \frac{k\alpha}{2\pi}\int_{\mathcal{M}} Tr\left(\omega\wedge\left(d\omega + \frac{2}{3}\,\omega\wedge\omega\right) + \frac{e}{l^2}\wedge T\right), \end{aligned} \tag{4.9}$$

[13] The definition depends on the signature of the internal metric η_{ab}. Our formulae are the case where the number of negative signatures is odd. For more details about my conventions, see Appendix **A**.

where $\Omega^{\pm} = \beta l \pm \alpha$, $Tr(J_a^{\pm} J_b^{\pm}) = (1/2)\eta_{ab}$ and

$$\begin{aligned} R &= d\omega + \omega \wedge \omega \\ &= \frac{1}{2} R^a{}_{b\nu\mu} dx^\nu \wedge dx^\mu \\ &= \frac{1}{2} e^a{}_\alpha e_b{}^\beta R^\alpha{}_{\beta\nu\mu} dx^\nu \wedge dx^\mu, \\ T &= de + 2\omega \wedge e \\ &= \frac{1}{2} T^a{}_{\nu\mu} dx^\nu \wedge dx^\mu \\ &= \frac{1}{2} e^a{}_\alpha T^\alpha{}_{\nu\mu} dx^\nu \wedge dx^\mu \end{aligned} \tag{4.10}$$

are the curvature and torsion 2-forms, respectively.

The equations of motion of the CS gravity, by treating A^+ and A^- "independently", become the usual forms

$$\begin{aligned} F^{\pm} &= dA^{\pm} + A^{\pm} \wedge A^{\pm} \\ &= R + \frac{1}{l^2} e \pm \frac{1}{l} T \wedge e = 0 \end{aligned} \tag{4.11}$$

or

$$T = 0, \tag{4.12}$$

$$R + \frac{1}{l^2} e \wedge e = 0, \tag{4.13}$$

where I have chosen the boundary conditions [54, 55, 57, 56], for each time slice,

$$A_0|_{\partial M} \propto A_\varphi|_{\partial M}, \tag{4.14}$$

$$\oint_{\partial M} dt d\varphi \, \langle A_\varphi, A_\varphi \rangle = \text{fixed} \tag{4.15}$$

with the boundary action

$$I_S = -\frac{k}{4\pi} \oint_{\partial M} dt d\varphi \, \langle A_\varphi, A_0 \rangle \,. \tag{4.16}$$

Here, I note that the equivalence of the equations (4.11) or (4.12, 4.13) and the Einstein equations (2.4) can be achieved only after solving the torsion-free condition (4.12) first. This should be the case since the spin-connections ω are not independent variables but are determined by the torsion-free condition (2.3) already. Actually, by plugging (4.12) into the action (4.9), it is a standard computation to show that (4.9) is equivalent to the gravity action (2.1), up to some boundary terms, with the couplings (see Appendix **A** for details)

$$k\beta = -\frac{1}{4G}, \quad \frac{\alpha}{l\beta} = \hat{\beta}. \tag{4.17}$$

But at this point, there is one subtlety here: The whole CS' equations of motion are not available when one of $\Omega^{\pm}$'s vanishes and this occurs with $\beta l = \alpha$ or $\hat{\beta} = 2$. In this critical case I have only one sector of the solutions in (4.11) such as the torsion-free condition (4.12) is not "necessarily " required. So, the equivalence of CS gravity (4.9) with the GCS-corrected gravity (2.1) can *not* be achieved in this case generally. However, if I restrict the solution space to the torsion-free ones only, the equivalence is admitted still. This is actually the situation that I consider in this paper since the BTZ solution (2.7) satisfies (4.12) and (4.13), which do not depend on the choice of ω or e.

Now, in order to study the black hole solution (2.7) in the context of the CS gravity, it is convenient to introduce a proper radial coordinate ρ such as (2.7) can be written as [14]

$$ds^2 = -\sinh^2\rho \left(\frac{r_+ dt}{l} - r_- d\varphi \right)^2 + l^2 d\rho^2 + \cosh^2\rho \left(\frac{r_- dt}{l} - r_+ d\varphi \right)^2 \tag{4.18}$$

with

$$r^2 = r_+^2 \cosh^2\rho - r_-^2 \sinh^2\rho. \tag{4.19}$$

In these coordinates, the (outer) event horizon is at $\rho = 0$ and hence this metric describes the exterior of the horizon for real values of ρ, but the interior for imaginary values of ρ. Then, it is easily checked that the 1-form gauge connections are given, in the proper coordinates, by

$$\begin{aligned} \mathbf{A}^{\pm 0} &= \pm \frac{r_+ \pm r_-}{l} \left(\frac{dt}{l} \mp d\varphi \right) \sinh \rho, \\ \mathbf{A}^{\pm 1} &= \pm d\rho, \\ \mathbf{A}^{\pm 2} &= \frac{r_+ \pm r_-}{l} \left(\frac{dt}{l} \mp d\varphi \right) \cosh \rho. \end{aligned} \tag{4.20}$$

[The superscript indices denote the group indices $a = 0, 1, 2$.] In matrix form [15], this becomes

$$\mathbf{A}^{\pm} = \frac{1}{2} \begin{pmatrix} \pm d\rho & z_\pm \; e^{\mp\rho} dx^{\pm} \\ z_\pm \; e^{\pm\rho} dx^{\pm} & \mp d\rho \end{pmatrix}, \tag{4.21}$$

where $z_\pm \equiv (r_+ \mp r_-)/l$ and $x^{\pm} = t/l \pm \varphi$. From this, the polar components [16] in the proper coordinates can be obtained as

$$A^{\pm}_{\rho} = \pm J_1, \; A^{\pm}_{\varphi} = \mp z_\pm \; (U^{-1} J_2 U), \; A^{\pm}_t = \mp l A^{\pm}_{\varphi} \tag{4.22}$$

[14]Note that the sign convention of φ differs from Ref. [54] such as it agrees with the original BTZ metric (2.7) [23, 24]. This agrees also with Refs. [8, 9, 10, 39].

[15]I take $J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\epsilon_{012} = 1$ as in Ref. [54]. The final results about the Virasoro algebras, however, do not depend on the choice of the representation.

[16]Here, $A_\rho = \hat{\rho}^i A_i, A_\varphi = \hat{\varphi}^i A_i$, for the orthogonal unit vectors $\hat{\rho}, \hat{\varphi}$ on the spatial boundary $\partial\Sigma$ with $\mathcal{M} = \Sigma \times R$; Σ is a 2-dimensional disc of space and R is a 1-dimensional infinite, real manifold of time.

with

$$U = \left(\begin{array}{cc} e^{\pm\rho/2} & 0 \\ 0 & e^{\mp\rho/2} \end{array} \right). \tag{4.23}$$

Here, I note that this solution satisfies the boundary conditions (4.14), (4.15) for *any* radius ρ such as the solution can be implemented even at the boundary whose radius may be arbitrary, from 0 (at r_+) to ∞. And the condition (4.15) implies the $micro-canonical$ ensemble from the relation $\left\langle A_\varphi^\pm, A_\varphi^\pm \right\rangle \sim (m \pm j/l)$.

C. The Symmetry algebras and classical central terms

The CS action has the gauge and diffeomorphism (*Diff*) symmetries. If there are boundaries, the central terms appear in the symmetry algebras even at the classical level.

For the time-independent gauge transformation [D_i is the covariant derivative $D_i^{ab} = \delta^{ab}\partial_i + \epsilon^{ab}{}_c A_i^c$]

$$\delta_\lambda A_i^{\pm a} = (D_i \lambda^\pm)^a, \quad \delta_\lambda A_0^{\pm a} = \epsilon^a{}_{bc} A_0^{\pm b} \lambda^{\pm c}, \tag{4.24}$$

the Lagrangian of (4.9) transforms as $\delta_\lambda L_{CS} = dX_\lambda^\pm/dt$ with $X_\lambda^\pm = (k\Omega^\pm/4\pi)\int_\Sigma d^2x\, \epsilon^{ij} Tr(\partial_i \lambda^\pm A_j^\pm)$. The conserved Noether charges for the right/left-moving CS sectors then become

$$\begin{aligned} Q^\pm(\lambda) &= \frac{k\Omega^\pm}{4\pi}\int_\Sigma d^2x\, \epsilon^{ij}\, Tr(F_{ij}^\pm \lambda^\pm) - \frac{k\Omega^\pm}{2\pi}\oint_{\partial\Sigma} d\varphi\, Tr(A_\varphi^\pm \lambda^\pm) \\ &\equiv Q_B^\pm(\lambda) + Q_S^\pm(\lambda), \end{aligned} \tag{4.25}$$

where $Q_B(\lambda)$ and $Q_S(\lambda)$ are the bulk and surface parts, respectively and $F_{ij} = F_{ij}^a J_a,\ F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + \epsilon^a{}_{bc} A_i^b A_j^c$.

The Noether charges satisfy the Kac-Moody algebras with the *classical* central terms in the Dirac bracket algebras

$$\begin{aligned} \{Q^\pm(\lambda), Q^\pm(\eta)\}^* &\approx \{Q_S^\pm(\lambda), Q_S^\pm(\eta)\}^* \\ &\approx Q_S^\pm([\lambda,\eta]) + \frac{k\Omega^\pm}{2\pi}\oint_{\partial\Sigma} d\varphi\, Tr(\lambda^\pm \partial_\varphi \eta^\pm), \end{aligned} \tag{4.26}$$

where $[\lambda,\eta]^a = \epsilon^a{}_{bc}\lambda^b\eta^c$. Here, the Dirac bracket is defined by

$$\{A,B\}^* = \{A,B\} - \int [du][dv]\{A, Q_B^\pm(u)\}\Delta^{-1}(u,v)\{Q_B^\pm(v), B\}, \tag{4.27}$$

where Δ^{-1} is defined as the *functional* inverse of

$$\Delta(\lambda^\pm, \eta^\pm) = \{Q_B^\pm(\lambda), Q_B^\pm(\eta)\}, \tag{4.28}$$

which depends, eventually, only on the functions λ, η, A_φ which live only on the boundary: $\int[du]\Delta(\lambda^\pm, u)\Delta^{-1}(u, \eta^\pm) = \int[du]\Delta^{-1}(\eta^\pm, u)\Delta(u, \lambda^\pm) = \delta(\lambda^\pm - \eta^\pm)$[17]. [The *weak* equality '$\approx$' means the equality up to the constraint $Q_B^\pm(\lambda) = 0$, which is the integrated form of the Gauss-law constraints $F_{ij}^\pm \approx 0$ with a *smearing* function λ.] This bracket satisfies $\{Q_B^\pm(\lambda), B\}^* \approx 0$ for any function(al) B and so the constraint $Q_B^\pm(\lambda) = 0$ can be imposed consistently in the Hamiltonian formulation. And it is important to note that the central terms are the results of the gauge symmetry breakings at the boundary, which make the gauge symmetry constraints be "second-class" in the Poisson brackets

$$\{Q_B^\pm(\lambda), Q_B^\pm(\eta)\} = Q_B^\pm([\lambda, \eta]) - \frac{k\Omega^\pm}{2\pi} \oint_{\partial\Sigma} d\varphi \; Tr(\lambda^\pm D_\varphi \eta^\pm), \tag{4.29}$$

as has been first computed in Ref. [57][18].

Moreover, for a spatial and time-independent *Diff*:

$$\begin{aligned} \delta_f x^\mu &= -\delta^\mu{}_k f^{\pm k}, \\ \delta_f A_i^{\pm a} &= f^{\pm k}\partial_k A_i^{\pm a} + (\partial_i f^{\pm k}) A_k^{\pm a}, \\ \delta_f A_0^{\pm a} &= f^{\pm k}\partial_k A_0^{\pm a}, \end{aligned} \tag{4.30}$$

the Lagrangian of (4.9) transforms as $\delta_f L_{CS} = dX_f^\pm/dt$ with $X_f^\pm = (k\Omega^\pm)/(4\pi) \oint_\Sigma d^2x \, Tr(f^{\pm\rho} A_\rho^\pm A_\varphi^\pm)$ when the boundary conditions "$A^{\pm a}{}_\rho|_{\partial\Sigma}$=constant" is imposed, which is a quite natural choice according to the explicit BTZ solution (4.22). Actually, this boundary condition is crucial for the existence of the central terms in the Virasoro algebras: Another boundary condition $f^{\pm\rho}|_{\partial\Sigma} = 0$ is also possible in order that there be *Diff* invariance, i.e., $\delta_{f^\pm} L_{CS} = dX/dt$ for the CS lagrangian L_{CS} and some function X, but in this case there is no classical central term [56, 57]; in Ref. [5], the "gauge condition" $A^{\pm a}{}_\rho|_{\partial\Sigma} = 0$ has been considered but these conditions are also too strong to admit the classical central terms and moreover this contradicts to the black hole solution (4.22) on the boundary.

[17] Since all the calculations involving Δ, Δ^{-1} are performed on the boundary $\partial\Sigma$, this rather formal definition works even though neither I confine to $\lambda, \eta, ..., etc.$ which live only on the boundary nor know the explicit form of Δ^{-1} [57]. This would be manifest in the Bergman-Komar's construction of Dirac brackets [67, 68]. (See Ref. [69] for a related discussion.) On the other hand, if I consider the matrix $\Delta(u, v)$ which is defined in the space of u, v which live only on the boundary, it is straightforward to compute Δ^{-1} (u, v are treated as the indices of the matrix) unless one considers a trivial (bulk) theory of $\Delta(u, v) = 0$, which is the $zero-mode$ [57, 48, 70, 71]; recently this approach has been also applied to the non-commutative open string and D-branes [72, 73, 74, 75, 76] to obtain the non-commutative open-string "coordinates" on the D-branes with a background B field.

[18] The mathematical origin of the central term is the modifications in the usual formulae for the derivatives of Dirac delta function $\int_\Sigma d^2x' \hat{r}'^i \partial_i' \delta^2(x - x')\eta(x') = -\hat{r}^i \partial_i \eta(x) + \delta(r - a)\eta(a, \varphi)$, $\int_\Sigma d^2x' [\epsilon^{ij} \partial_i \partial_j' \delta^2(x - x')]\eta(x') = -\delta(r - a)\hat{\varphi}^i \partial_i \eta(a, \varphi)$ due to non-vanishing test function η on a boundary at $r = a$. It seems that this novel boundary effects have not been well recognized in the earlier works though [54, 77]; actually it is misleading to introduce the gauge fixing conditions as in Ref. [54] to obtain the Dirac bracket algebras (4.26).

Then, the conserved Noether charges become

$$
\begin{aligned}
Q^{\pm}(f) &= \frac{k\Omega^{\pm}}{4\pi}\int_{\Sigma} d^2x\ Tr(f^{\pm k}A_k^{\pm}\epsilon^{ij}F_{ij}^{\pm}) \\
&\quad -\frac{k\Omega^{\pm}}{4\pi}\oint_{\partial\Sigma} d\varphi\ Tr(2f^{\pm\rho}A_{\rho}^{\pm}A_{\varphi}^{\pm}+f^{\pm\varphi}A_{\varphi}^{\pm}A_{\varphi}^{\pm}+f^{\pm\varphi}A_{\rho}^{\pm}A_{\rho}^{\pm}) \\
&\equiv Q_B^{\pm}(f)+Q_S^{\pm}(f)
\end{aligned}
\tag{4.31}
$$

with the bulk and boundary parts $Q_B^{\pm}(f)$, $Q_S^{\pm}(f)$, respectively as in (4.25); the last constant term, proportional to $Tr(A_\rho A_\rho)$, in (4.31) was included to obtain the *standard* Virasoro central term with the help of the ambiguities in the definition of Noether charge. These satisfy the Virasoro algebras with the *classical* central terms in the Dirac bracket algebras[19]

$$
\begin{aligned}
\{Q^{\pm}(f),Q^{\pm}(g)\}^* &\approx \{Q_S^{\pm}(f),Q_S^{\pm}(g)\}^* \\
&\approx Q_S([f,g])-\frac{k\Omega^{\pm}}{2\pi}\ Tr(A_\rho^{\pm}A_\rho^{\pm})\oint_{\partial\Sigma} d\varphi(f^{\pm\rho}\partial_\varphi g^{\pm\rho}-f^{\pm\varphi}\partial_\varphi g^{\pm\varphi}),
\end{aligned}
\tag{4.32}
$$

where $[f,g]^k=f^\varphi\partial_\varphi g^k-g^\varphi\partial_\varphi f^k$ is Lie bracket on the boundary circle ($\partial\Sigma$).

Here I note that the Virasoro algebras (4.32) are the results of the Kac-Moody algebras (4.26), but *the latter do not imply the former always* since the former depend crucially on the boundary conditions. This can be also understood from the "qualitative" difference of X_f for *Diff* with X_λ for the gauge transformation in that there are only the boundary contributions in the former whereas there are the bulk as well as boundary contributions in the latter for the general gauge group; they can be equivalent up to the Gauss-law constraint $F_{ij}\approx 0$ for the $U(1)$ group, but this is not relevant to our BTZ system.

I also note that the algebras (4.32) satisfies the Jacobi identity only for the subset of transformation with $f^\rho|_{\partial\Sigma}\propto\partial_\varphi f^\varphi|_{\partial\Sigma}$ and $g^\rho|_{\partial\Sigma}\propto\partial_\varphi g^\varphi|_{\partial\Sigma}$ such that only the third- and first-order derivatives appear in the central terms [55]. This particular form corresponds to the *Diff* which deforms *across* the boundary with proportionality to the steepness ($\partial_\varphi f^\varphi$) of *Diff* along the circle ($\partial\Sigma$); the boundary $\partial\Sigma$ responds as an *elastic medium* to the deformations [56][20].

D. Asymptotic isometries and the central charges

Under the *Diff* generated by the Noether charges $Q^{\pm}(f)$, the gauge fields of (4.22), repre-

[19] In the usual context of the Regge-Teitelboim method [54, 78, 79, 80, 81, 82], the Dirac brackets of (4.26) and (4.32) are computed from the equivalence of the Dirac brackets and Poisson brackets $\{Q(f),Q(g)\}^*\approx\{Q(f),Q(g)\}$, which was *assumed* implicitly or explicitly, in models with the *finite* as well as the infinite boundaries. But in the CS approach one can explicitly *prove* the equivalence from the fact of $\{Q,Q_B\}=0$ for *any* radius [57, 83].

[20] This seems to imply a *complementary* between the black hole and elastic medium of the boundary which looks similar to the complementary of black hole and strings on the *stretched* horizon [84, 85, 86, 87].

senting the BTZ black hole, have the transformations

$$\begin{aligned} \delta_f A^{\pm}_{\varphi} &= \frac{1}{2}\begin{pmatrix} \pm\partial_\varphi f^{\pm\rho} & z_{\pm}\ e^{\mp\rho}(f^{\pm\rho}\mp\partial_\varphi f^{\pm\varphi}) \\ -z_{\pm}\ e^{\pm\rho}(f^{\pm\rho}\pm\partial_\varphi f^{\pm\varphi}) & \mp\partial_\varphi f^{\pm\rho} \end{pmatrix}, \\ \delta_f A^{\pm}_{\rho} &= 0. \end{aligned} \tag{4.33}$$

This implies that the black hole solution (4.22) admits the isometries, i.e., $\delta_f A^{\pm}_i = 0$ as $\rho \to \infty$ when

$$f^{\pm\rho}|_{\partial\Sigma} = -\partial_\varphi f^{\pm\varphi}|_{\partial\Sigma} \tag{4.34}$$

is satisfied, though *not* necessarily for arbitrary ρ. This exactly agrees, to the leading order, with the asymptotic isometries found by Brown-Henneaux [53][21]. Contrary to the existence of the central term itself, this result is a purely non-Abelian effect which comes from the off-diagonal parts.

Now, by substituting (4.34) with the insertion of $Tr(A^{\pm}_{\rho}A^{\pm}_{\rho}) = 1/2$ for the black hole solution (4.22), the algebras (4.32) become the standard Virasoro algebras, in the coordinate space, with *classical* central charges

$$c^{\pm} = -12k\Omega^{\pm}Tr(A^{\pm}_{\rho}A^{\pm}_{\rho}) = \gamma^{\pm}\frac{3l}{2G} \tag{4.35}$$

with $\gamma^{\pm} = 1 \pm \hat{\beta}$. In the $\hat{\beta} \to 0$ limit, these classical central charges reduce to the usual result of Brown-Henneaux [53] for the asymptotic isometry of AdS_3 and also agrees exactly with that of conformal anomaly computation [37, 38, 39]. But interestingly, the $\hat{\beta}$-dependent central charges give an exact agreement also with the $semi-classical$ central charges

$$\hat{c}^{\pm} = \frac{c^{\pm}}{\hbar}, \tag{4.36}$$

as in (4.2), that have been obtained from gravitational anomaly computation; this seems to be a quite non-trivial result since I don't see any general proof about the equivalence of the two central charges even without the GCS term though it seems to be quite plausible in the context of AdS/CFT correspondence, which identifies the "classical" asymptotic CFT of AdS on the one hand with the "quantum" CFT on the boundary on the other hand.

The more familiar momentum-space Virasoro algebras (4.1) can be obtained by defining the boundary parts of the Noether charges in (4.31) as

$$Q^{\pm}_S(f) \equiv \frac{1}{2\pi}\oint_{\partial\Sigma} d\varphi f^{\pm\varphi}\left(\sum_n L^{\pm}_n e^{+in\varphi}\right) \tag{4.37}$$

[21] There are several other ways to implement the *Diff* even for the finite values of ρ [88], where there are some $RG-flows$ of the central charges and conformal weights without changing the statistical entropies. So, there remains the question on the very place where the black hole's degrees of freedom live.

and the central charges are given by (4.35). The ground state generators, from the definition, become

$$L_0^{\pm} = -\frac{k\Omega^{\pm}}{4\pi}\oint_{\partial\Sigma} d\varphi\ Tr(A_\varphi^{\pm}A_\varphi^{\pm} + A_\rho^{\pm}A_\rho^{\pm}) = \gamma^{\pm}\frac{1}{2}(lm \pm j) + \frac{c^{\pm}}{24}. \tag{4.38}$$

Note that the $\hat{\beta}$-dependent terms, as well as $\hat{\beta}$-independent terms, agree exactly with $\hat{L}_0^{\pm} = L^{\pm}/\hbar$ of (3.6). So, if I define the black hole's mass and angular momentum *canonically* as in (3.6), from the general consideration of conformal field theory on the torus [89],

$$L_0^{\pm} = \frac{lM \pm J}{2} + \frac{c^{\pm}}{24} \tag{4.39}$$

one obtains the same mass and angular momentum as in the anomaly approach [8, 9], which agree with the usual ADM quantities of (2.10) [6, 25, 26] also. It does not seem that this is not just a coincidence but there be some deep reasons involving the *holographic* principle; but our CFT computation of the statistical entropy does not depend on the manners of identifications of M and J but only on the geometrical quantities of r_+ and r_- such as the CFT computation provides a quite independent estimation of the *would-be* black hole entropy.

Now with the Virasoro algebras with "classical" data of the central charges (4.35) and the ground state generator $L_0^{\pm}$ in (4.38), it is straightforward to obtain the corresponding quantum Virasoro algebras [44]: If I consider the canonical quantization

$$[\mathbf{L}_m^{\pm}, \mathbf{L}_n^{\pm}] = i\hbar\{L_m^{\pm}, L_n^{\pm}\}^* \tag{4.40}$$

for the quantum operators $\mathbf{L}_m^{\pm}$ and a rescaling transformation

$$\mathbf{L}_m^{\pm} \rightarrow \hbar(:\hat{L}_m^{\pm}: + \hbar a^{\pm}\delta_{m,0}) \tag{4.41}$$

for the normal ordered operators $:\hat{L}_m^{\pm}:$ with some possible normal ordering constants $a^{\pm}$, one can easily find the corresponding quantum Virasoro algebras

$$[:\hat{L}_m^{\pm}:, :\hat{L}_n^{\pm}:] = (m-n):\hat{L}_{m+n}^{\pm}: + \frac{\hat{c}_{\rm tot}^{\pm}}{12}m(m^2-1)\delta_{m,-n} \tag{4.42}$$

with

$$\hat{c}_{\rm tot}^{\pm} = \frac{c^{\pm}}{\hbar} + c_{\rm quant}^{\pm}. \tag{4.43}$$

Here, the quantum correction $c_{\rm quant}^{\pm}$ is due to some operator re-orderings and it is order of $O(1)$.

With the Virasoro algebras of $:\hat{L}_m:$ in the standard form, which is defined on the plane, one can use the Cardy formula for the asymptotic states [44, 45, 46] as in (3.7)

$$\log\rho(\hat{\Delta}^{\pm}) \simeq 2\pi\sqrt{\frac{1}{6}\left(\hat{c}_{\rm tot}^{\pm} - 24\hat{\Delta}_{\rm min}^{\pm}\right)\left(\hat{\Delta}^{\pm} - \frac{\hat{c}_{\rm tot}^{\pm}}{24}\right)}, \tag{4.44}$$

where $\hat{\Delta}^\pm$ are the eigenvalues of : $\hat{L}_0^\pm$: for the black-hole quantum states $|\hat{\Delta}^\pm\rangle$ and $\hat{\Delta}^\pm_{\rm min}$ are their minimum values. When expressed in terms of the classical generators $L_0^\pm$ and the central charges $c^\pm$ through

$$\hat{\Delta}^\pm = \frac{L_0^\pm}{\hbar} - \hbar a^\pm, \tag{4.45}$$

one obtains

$$\log\rho(L_0^\pm) \simeq \frac{2\pi}{\hbar}\sqrt{\frac{1}{6}\left(c^\pm - 24L^\pm_{0\ {\rm min}} + \hbar c^\pm_{\rm quant} + 24\hbar^2 a^\pm\right)\left(L_0^\pm - \frac{c^\pm}{24} - \frac{\hbar c^\pm_{\rm quant}}{24} - \hbar^2 a^\pm\right)}. \tag{4.46}$$

This approach shows explicitly how the *classical* Virasoro generators $L_0^\pm$ and central charges $c^\pm$ can give the correct order of the semiclassical BH entropy (2.15),

$$S_{\rm BH} \simeq \frac{\mathcal{A}_+}{4\hbar G} \tag{4.47}$$

with $\sqrt{c^\pm L_0^\pm} \sim \mathcal{A}_+/G$; the quantum corrections due to reordering give the negligible order of $O(1)$ effect to the entropy when one considers the large black holes with $\mathcal{A}_+/(G\hbar) \gg 1$.

Then, the statistical entropy for the asymptotic states becomes [omitting the small quantum corrections of the order of $O(1)$]

$$\begin{aligned} S_{\rm stat} &= \log\,\rho(L_0^+) + \log\,\rho(L_0^-) \\ &= \frac{\pi}{4G\hbar}(|\gamma^+| + |\gamma^-|)r_+ + \frac{\pi}{4G\hbar}(|\gamma^+| - |\gamma^-|)r_- \ , \end{aligned} \tag{4.48}$$

where I have chosen $L^\pm_{0\ ({\rm min})} = 0$, which corresponds to the AdS_3 vacuum solution where $m = -1/(8G)$ and $j = 0$ in agreement with (3.10). This has exact matchings with (3.10) in the $\hat{\beta}$-dependent correction terms as well as $\hat{\beta}$-independent terms. I note also that the "$1/\hbar$" factor in the black hole entropy (4.48) was generated in the process of canonical quantization of the classical Virasoro algebras and this $\hbar$-generating mechanism differs from that of *Euclidean* action approaches in Ref. [90].

So, the statistical entropy, based on the classical symmetry algebras, agrees with the thermodynamic black hole entropy even in the correction terms due to the GCS term, as well as the usual one for the Einstein-Hilbert action[22]. This might a subtle issue because of some normalization differences between the different bases and conventions in the literatures: Actually there are ubiquitous factor "2" differences between different bases. So, I have included some details about the transformations of the formulae between the different bases and conventions

[22]The usual Brown-Henneaux's approach would produce the same factor matching since this is also based on the *same* classical symmetry algebras basically on the *physical* subspace where all the physical constraints are strongly imposed [64].

in the Appendix **A** in order to ensure that this exact factor matching is a *solid* result actually.

V. Summary and discussions

I have studied the thermodynamics of BTZ black hole in the presence of the higher-derivative corrections of the gravitational Chern-Simons term and its solid connection with the statistical approaches based on the holographic anomalies and also the classical symmetry algebras.

The main results are as follows:

First, for the case of large coupling $|\hat{\beta}| > 1$ the new entropy formula which requires rather unusual characteristic temperature $T_- = \kappa/(2\pi)|_{r_-}$, which is negative-valued, (for $\hat{\beta} > 1$) or $T_-{}' = -T_-$ (for $\hat{\beta} < -1$), and angular velocity Ω_-, which is the inner-horizon angular velocity in BTZ, is proposed from the purely thermodynamic point of view such as the second law of thermodynamics be guaranteed.

Second, I have found strong supports of the proposal from the CFT based approaches which reproduce the new entropy formulae for $|\hat{\beta}| > 1$ as well as the usual entropy formula for the small coupling $|\hat{\beta}| \leq 1$.

Third, I have found the exact "factor" matchings between the holographic anomaly approach, which is intrinsically a "quantum" approach, and the classical symmetry algebra approach from the Chern-Simons formulation of the three-dimensional gravity and this would provide a non-trivial check of the AdS/CFT-correspondence in the presence of higher-derivative terms in the gravity action.

Finally, as a by product, I have clarified how the correct "$1/\hbar$" factor in the semiclassical black hole entropy can be reproduced from the appropriate recovering of $\hbar$, which is hidden in the usual anomaly computations.

Now, several comments are in order.

1. *On the general validity of the Cardy formula even with higher-derivative/curvature corrections* : It is interesting to note that the statistical entropy (3.10) from the Cardy formula (3.7) has basically the *same form* for both the Einstein-Hilbert action and the GCS corrected action; *the only changes are some correction terms in the central charges and the conformal weights themselves rather than considering the higher-order corrections to the Cardy formula as in Ref. [46].* This seems to be true even in the presence of higher-"curvature" terms [59, 7, 10] and also in the supersymmetric black holes [91]. So there should be some explanations about this and actually this is the case. This comes from the fact that the higher-derivative/curvature actions do *not* necessarily imply the quantum corrections though the converse can be true [92, 93, 94]. So, if the higher-derivative/curvature gravities are treated semiclassically by neglecting the back-reaction effects, which are quantum effects, such as (3.9) or (3.13) is satisfied, the saddle-

point approximation for the Cardy formula (3.7) and so the entropy formula (3.10) are good approximations even with the higher-derivative/curvature terms in the gravity action [46]. There is another factor whose departure from unity is order of $O[\exp\{-2\pi\hat{\Delta}^{\pm}_{\rm eff}(\hat{\Delta}^{\pm}-\hat{\Delta}^{\pm}_{\rm min})/\hat{c}_{\rm eff}\}]$, but this correction, if there is, is not comparable with the leading term (3.10) and other higher-order corrections by departing the semi-classical limit of (3.13); in our case of the GCS-BTZ black holes there is already the corrections of order of $O(r_-/r_+)$ in the leading entropy (3.10), but this dominates the exponentially suppressed corrections. Hence *the leading Cardy formula (3.7) would have quite general validity for any kinds of semiclassical black holes in the higher-derivative/curvature gravities unless the condition (3.8) or (3.12) is violated.*

2. *Higher-order corrections to the saddle-point approximation*: By relaxing the semiclassical condition of (3.9) or (3.13) but keeping only the condition (3.8) or (3.12), the higher order corrections in the Cardy formula (3.7) can be evaluated by the steepest descent method, known as the Rademacher expansion [95, 96]. The statistical entropy then becomes, up to fourth order, (see Appendix **B** for the details) from (B.12)

$$\begin{aligned} S_{\rm stat(4)} &= (S_0^+ + S_0^-) - \frac{3}{2}\log(S_0^+ S_0^-) + \log(\hat{c}^+_{\rm eff}\hat{c}^-_{\rm eff}) + \log(\pi^3/18) - \frac{3}{8}\left(\frac{1}{S_0^+}+\frac{1}{S_0^-}\right) + O((S^{\pm})^{-2}) \\ &= S_{\rm stat} - \frac{3}{2}\log\left(\left(\frac{\pi}{G\hbar}\right)^2 |\gamma^+\gamma^-|(r_+^2-r_-^2)\right) + \log\left(\gamma^+\gamma^-\left(\frac{3l}{2G\hbar}\right)^2\right) + \log(\pi^3/18) \\ &\quad -\frac{3}{8}\left(\frac{G\hbar}{\pi}\right)^2 \frac{S_{\rm stat}}{|\gamma^+\gamma^-|(r_+^2-r_-^2)} + O((G\hbar)^2/r_+^2,\ (G\hbar)^2/r_-^2), \end{aligned} \tag{5.1}$$

where $S_0^{\pm}$ denote the right/left-moving parts of the leading entropy formula (3.10), i.e., $S_0^{\pm} = \log\rho(\hat{\Delta}^{\pm})$ with $S_0^+ + S_0^- = S_{\rm stat}$ and this is the expansion about the Planck constant $\hbar$. It would be a challenging problem to compute the loop-corrected black hole entropies in the gravity side also and compare with the above CFT result (5.1). Actually the loop corrections in the gravity side would not be trivial in this case since there would be now some propagating mode(s) with the GCS term [1, 2, 3, 97], in contrast to the usual BTZ black hole [46, 98].

$2\frac{1}{2}$. *Subtleties of extremal and near-extremal black holes* : If I consider extremal bare black holes with $r_+ = r_-$, i.e., $\hat{\Delta}^-_{\rm eff} = 0$, which saturates the mass bound $m = j/l$ and has vanishing temperatures, there seem to exist some subtleties in the above manipulations. Namely, the condition (3.13) does not apply and the back-reaction effect would not be negligible anymore in this case, such as I would need to consider "infinite" higher-order corrections in the steepest-descent approximations, which seems to be highly divergent from (5.1); other infinite series of exponential correction terms are actually of the form $O[(\hat{\Delta}^{\pm}-\hat{\Delta}^{\pm}_{\rm min})^m(\hat{\Delta}^{\pm}_{\rm eff}/\hat{c}_{\rm eff})^n\exp\{-2\pi\hat{\Delta}^{\pm}_{\rm eff}(\hat{\Delta}^{\pm}-\hat{\Delta}^{\pm}_{\rm min})/\hat{c}_{\rm eff}\}]$ with some positive integers m and n [46] such as the problematic part does not contribute further. But, actually this is not quite correct as can be seen easily in the original partition function (B.1). In the case of extremal bare black holes, the left-moving sector is absent in the partition function because of $\hat{L}_0^- - \hat{c}^-/24 = 0$ such as total partition function is given by, from (B.12),

$$\begin{aligned} S_{\mathrm{stat}(4):\ \mathrm{extreme}} &= \frac{2\pi r_+}{4G\hbar}|\gamma^+| - \frac{3}{2}\log\left(\frac{2\pi r_+}{4G\hbar}|\gamma^+|\right) + \log\left(\gamma^+\frac{3l}{2G\hbar}\right) + \frac{1}{2}\log(\pi^3/18) \\ &\quad -\frac{3}{8}\left(\frac{4G\hbar}{2\pi r_+}\right)\frac{1}{|\gamma^+|} + O((G\hbar)^2/r_+^2). \end{aligned} \tag{5.2}$$

This gives the correct BH entropy for the leading term as can be also read from (5.1) and there is no divergence in each order[23]. This implies that in the "near-extremal" case, the naive divergence in each term of (5.1) would cancel each other and one would have only some finite entropy. Actually this seems to be supported also by the exact Raedmacher expansion which shows that the *exact* entropy with all higher-order corrections is bounded by, up to some exponentially suppressed terms, the BH entropy, i.e., $0 \leq S_{\mathrm{exact}} < S_{BH}$ [99]; if there are no cancelations, the exact entropy S_{exact} would easily violate the above Birmingham-Sen's bound. On the other hand, it is important to note that the condition for the right-moving sector *only* can be satisfied, though not possible for the left-moving sector, such as the extremal bare black hole with vanishing temperature does *not* always imply the necessity of the higher-order corrections; but its relevance to the back-reaction effect is not clear [47].

On the other hand, the case of critical coupling $|\hat{\beta}| = 1$, which has the extremal bound $M^2 = J^2/l^2$ but *non-vanishing* temperature, has similar subtleties. In this case, one of $\gamma^\pm$ vanishes such as the condition (3.8) would be ambiguous even though overall $\gamma^\pm$ factor can be canceled for non-vanishing $\gamma^\pm$. And the condition (3.9) can not be satisfied either such as its entropy has similar divergence problem from (5.1) as in the bare extremal black holes. The resolution is similar to the bare-extremal black hole and the appropriate statistical entropies are given by, from (B.12),

$$\begin{aligned} S_{\mathrm{stat}(4):\ \hat{\beta}=\pm 1} &= \frac{2\pi(r_+ \pm r_-)}{4G\hbar} - \frac{3}{2}\log\left(\frac{2\pi(r_+ + r_-)}{4G\hbar}\right) + \log\left(2\frac{3l}{2G\hbar}\right) + \frac{1}{2}\log(\pi^3/18) \\ &\quad -\frac{3}{8}\left(\frac{4G\hbar}{2\pi(r_+ + r_-)}\right) + O((G\hbar)^2/(r_+^2 \pm r_-)^2) \end{aligned} \tag{5.3}$$

for $\hat{\beta} = \pm 1$ and these agree with the entropies (3.17, 3.18) in the leading order. But, if I consider the extremal bare black holes further with $r_+ = r_-$, the entropy for $\hat{\beta} = 1$ case reduces

[23]Interestingly, the factor "3/2" in the logarithmic term agrees with the corresponding corrections in the induced WZW model at the horizon within the context of CS gravity, in contrast to the factor "2" mismatches in the non-extremal black holes [46]. But it is subtle to compare with the purely gravity manipulation since there is no clear way to resolve a similar divergence problem.

to (5.2) whereas that for $\hat{\beta} = -1$ case has divergent higher order terms with the vanishing entropy in the leading term. This subtleties can be resolved again in the original partition function language; there the right-moving sector is absent, i.e., $L_0^+ - c^+/24 = 0$ due to $\gamma^+ = 1$, whereas the left-moving sector is also absent, i.e., $L_0^- - c^-/24 = 0$ due to $r_+ = r_-$ such as one has only a *single ground* state with $\rho(\hat{\Delta}^+, \hat{\Delta}^-) = 1$; this system satisfies the Nernst formulation of the third law of thermodynamics [49, 50, 51], i.e., $S_{\rm stat} = \log \rho = 0$, to *all orders* !

3. *Probing inside the outer horizon by the GCS action ?*: Although there are some solid supports from the second law of thermodynamics and the CFT approaches, the inner horizon's data, which are required in the complete formulae, look strange still; of course, the necessity of the inner horizon's data seems to be a quite general feature with quantum corrections from the result of (5.1) but the problem is that it occurs even at the leading, classical level. Actually, this would be much strange in the Euclidean method of conical singularity [9] or in the Wald's approach to compute the black hole entropy which gives the same entropy formula with the inner-horizon term even though it is given by some integrals over the outer horizon [9, 100, 101]. Furthermore, in the strong coupling of $|\hat{\beta}| > 1$, the inner horizon's temperature T_- or $T_-{}'$, and angular velocity Ω_- are needed in order to characterize the system. Even though the negative-valued temperature T_- for $\hat{\beta} > 1$ may be understood from the analogy with the spin systems, the obvious physical question would be the *observational* meaning of T_- and Ω_- by some physical processes. So, understanding the roles of the inner horizon's data appearing in the thermodynamics relations would be a challenging problem[24]; some possible probing, in the context of the AdS/CFT, beyond the event horizon have been considered recently [30, 31, 102], but this need further studies.

4. *Classical (in)stability of the $|\hat{\beta}| > 1$ black holes*: For the large coupling of $|\hat{\beta}| > 1$, the black-hole angular momentum is greater than its mass $J^2/l^2 \geq M^2$ and there are three known cases which show this "exotic" property, including the GCS case, in $D = 3$ and 5 [16, 17, 18, 19, 20]. There are no similar black hole solutions in $D = 2$ and 4, as far as I know. In $D \geq 6$ the "ultra-spinning" black holes are possible in the Einstein gravity [103] but it seems that there is a *classical instability* under small perturbations [104, 105]. So, it would be interesting to investigate this classical (in)stability in our exotic cases also; there might exist some topological reasons, but it is not clear in our case since there are propagating modes [97], in contrast to the ordinary BTZ black hole and KdS_3 solution [56].

[24] The probing of the inner horizon has been first remarked in Ref. [9].

Appendix A. Conventions and some useful formulae in differential forms

In this appendix, I summarize the conventions and some useful formulae in differential forms used in this paper. I have also included some details about the computations in order to ensure that the *exact* factor matching, which is directly related to the relation in (4.17), does not come from some normalization differences between different bases but a quite solid result. I have used the Lorentzian metric for the internal Lorentz indices $\eta_{ab} = diag(-1, 1, 1)$ and $\epsilon_{012} = -\epsilon^{012} = 1$. [For the s-negative signatures in the metric generally, a number of formulae will contain the factor of $(-1)^s$ [106, 97, 107].]

The invariant quadratic forms for the $SL(2, \mathbf{R})$ generators are (4.7), i.e.,

$$\left\langle J_a^{\pm}, J_b^{\pm} \right\rangle = \frac{1}{2} (\alpha \pm \beta l)\eta_{ab}, \quad \left\langle J_a^{\pm}, J_b^{\mp} \right\rangle = 0. \tag{A.1}$$

and the Lorentz indices are raised and lowered by the metric η_{ab}. [One can consider conveniently the invariant form as $\left\langle J_a^{\pm}, J_b^{\pm} \right\rangle = (\alpha \pm \beta l) Tr(J_a^{\pm} J_b^{\pm})$ by considering the explicit matrix representation of the generators with $Tr(J_a^{\pm} J_b^{\pm}) = (1/2)\eta_{ab}$ as in the Sec. IV, but the final results do not depend on the representations; so I will keep (A.1) in this appendix.]

Now let me prove (4.9), (4.11) and the relations in (4.17). To this end, I first note that the CS action in (4.9) can be written, in the component form for the internal space, as

$$\frac{4\pi}{k} I_{CS}[\mathbf{A}] = \int_{\mathcal{M}} \frac{1}{2}\, \Omega^+ \left(\eta_{ab} A^{+a} \wedge dA^{+b} + \frac{1}{3}\, \epsilon_{abc} A^{+a} \wedge A^{+b} \wedge A^{+c} \right) \; - \; (+ \leftrightarrow -)\,, \tag{A.2}$$

where I have used

$$\begin{aligned}
\langle \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \rangle &= \left\langle A^+ \wedge \frac{1}{2}[A^+, A^+] \right\rangle \; + \; (+ \leftrightarrow -) \\
&= \frac{1}{2} A^{+a} \wedge A^{+b} \wedge A^{+d} \epsilon_{ab}{}^{c} \left\langle J_d^+, J_c^+ \right\rangle + (+ \leftrightarrow -) \\
&= \frac{1}{2} A^{+a} \wedge A^{+b} \wedge A^{+c} \cdot \frac{1}{2} \epsilon_{abc} (\alpha + \beta l) + \frac{1}{2} A^{-a} \wedge A^{-b} \wedge A^{-c} \cdot \frac{1}{2} \epsilon_{abc} (\alpha - \beta l) \\
&= \frac{1}{4} \Omega^+ \epsilon_{abc} A^{+a} \wedge A^{+b} \wedge A^{+c} - \frac{1}{4} \Omega^- \epsilon_{abc} A^{-a} \wedge A^{-b} \wedge A^{-c}
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
\langle \mathbf{A} \wedge d\mathbf{A} \rangle &= \left\langle A^+ \wedge dA^+ \right\rangle + (+ \leftrightarrow -) \\
&= A^{+a} \wedge dA^{+b} \left\langle J_a^+, J_b^+ \right\rangle + (+ \leftrightarrow -) \\
&= A^{+a} \wedge dA^{+b} \cdot \frac{1}{2} (\alpha + \beta l) \eta_{ab} + A^{-a} \wedge dA^{-b} \cdot \frac{1}{2} (\alpha - \beta l) \eta_{ab} \\
&= \frac{1}{2} \Omega^+ \eta_{ab} A^{+a} \wedge dA^{+b} - \frac{1}{2} \Omega^- \eta_{ab} A^{-a} \wedge dA^{-b}
\end{aligned} \tag{A.4}$$

with the one-form gauge fields $\mathbf{A} = A^{+a} J_a^+ + A^{-a} J_a^-$ and $\Omega^\pm = \beta l \pm \alpha$.

By considering $A^{\pm a} = \omega^a \pm e^a/l$ with the spin connections ω^a and the triads e^a, one can find that the CS action (A.2) becomes, after some manipulations,

$$\begin{aligned}
\frac{4\pi}{k} I_{CS} &= \int_{\mathcal{M}} \Big[\alpha \omega^a \wedge \Big(d\omega_a + \frac{1}{3}\, \epsilon_{abc} \omega^b \wedge \omega^c \Big) + \frac{\alpha}{l^2} e^a \wedge (de_a + \epsilon_{abc} \omega^b \wedge e^c) \\
&\quad + \beta e^a \wedge \Big(2d\omega_a + \epsilon_{abc}\omega^b \wedge \omega^c + \frac{1}{3l^2}\, \epsilon_{abc} e^b \wedge e^c \Big) - \beta d(\omega^a \wedge e_a) \Big] \\
&= \alpha \int_{\mathcal{M}} \Big[\omega^a \wedge \Big(d\omega_a + \frac{1}{3}\, \epsilon_{abc}\omega^b \wedge \omega^c \Big) + \frac{1}{l^2} e^a \wedge T_a \Big] \\
&\quad + \beta \int_{\mathcal{M}} \Big(2e^a \wedge R_a + \frac{1}{3l^2}\, \epsilon_{abc} e^a \wedge e^b \wedge e^c \Big) - \beta \oint_{\partial\mathcal{M}} \omega^a \wedge e_a, \qquad \text{(A.5)}
\end{aligned}$$

where I have defined the curvature two-form, in *vector* form basis,

$$\begin{aligned}
R^a &= \frac{1}{2}\, \epsilon^a{}_{bc} R^{bc} \\
&= d\omega_a + \frac{1}{2}\, \epsilon_{abc} \omega^b \wedge \omega^c \qquad \text{(A.6)}
\end{aligned}$$

from $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ and $\omega_{ab} = -\epsilon_{abc}\omega^c,\ \omega^a = (1/2)\epsilon^{abc}\omega_{bc}$ [note the difference in the numerical factors of the quadratic terms in (A.6) and the bracket of the first term in the final result of (A.5) such as the latter can not be expressed as R^a only]. The negative sign comes from $(-1)^s$ factor when we consider $\epsilon_{abc}\epsilon^{ade} = (-1)^s(\delta^d_b \delta^e_c - \delta^e_b \delta^d_c)$ for s negative signatures in the metric η_{ab}. This becomes (4.9) in the compact form notation with the trace Tr, up to the boundary term–actually this becomes a "half" of the Gibbons-Hawking's boundary term $2\oint_{\mathcal{M}} K$, for the extrinsic curvature scalar K of the boundary, in the gravity action [108, 109]. Note also that there are factor "2" difference in the triple wedge products of ω's between (4.9) and (A.5).

Now, in order to determine the coefficients α, β, I need to compare the result (A.5) in the *vector* basis with that of the usual *tensor* form basis. To this end, I first note that

$$\begin{aligned}
I_1 \equiv \int 2e^a \wedge R_a &= \int \epsilon_{abc} e^a \wedge R^{bc} \\
&= \int \epsilon_{abc} e^a_\mu \cdot \frac{1}{2}\, R^{bc}{}_{\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \\
&= \frac{1}{2} \int d^3x\ \epsilon^{\mu\nu\rho} \epsilon_{abc} e^a{}_\mu R^{bc}{}_{\nu\rho} \\
&= \frac{1}{2} \int d^3x\ \epsilon^{\mu\nu\rho} \epsilon_{abc} e^a{}_\mu e^a{}_\alpha e^a{}_\beta R^{\alpha\beta}{}_{\nu\rho} \\
&= \frac{1}{2} \int d^3x \sqrt{-g}\ \epsilon^{\mu\nu\rho} \epsilon_{\alpha\beta\mu} R^{\alpha\beta}{}_{\nu\rho} \\
&= -\int d^3x \sqrt{-g}\ R, \qquad \text{(A.7)}
\end{aligned}$$

where I have denoted $R_{bc\nu\rho} = \partial_\nu \omega_{bc\nu} + \omega^b{}_{d\nu}\omega^{dc\rho} - (\nu \leftrightarrow \rho)$ in the second line and I have used $dx^\mu \wedge dx^\nu \wedge dx^\rho = \epsilon^{\mu\nu\rho}\ d^3x$ in the third line; $\epsilon_{abc}e^a{}_\mu e^a{}_\alpha e^a{}_\beta = e\epsilon_{\mu\alpha\beta}$ with $e = \sqrt{-g}$ [e is the determinant of $e^a{}_\mu$] due to $g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu$ in the fourth line; the negative sign in the final line comes from $(-1)^s$ factor with $s = 1$. This is the usual Einstein-Hilbert action, up to the sign.

Similarly, one can show that

$$\begin{aligned} I_2 \equiv \int \frac{1}{3l^2}\ \epsilon_{abc}e^a \wedge e^b \wedge e^b &= \int d^3x\ \frac{1}{3l^2}\ \epsilon^{\mu\nu\rho}\epsilon_{abc}e^a{}_\mu \wedge e^b{}_\nu \wedge e^b{}_\rho \\ &= \int d^3x\sqrt{-g}\ \frac{1}{3l^2}\ \epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\rho} \\ &= -\int d^3x\sqrt{-g}\ \frac{2}{l^2}, \end{aligned} \tag{A.8}$$

where I have used $\epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\rho} = (-1)^s 3!$ in the final line. This is the cosmological constant action.

Next, I note that

$$\begin{aligned} I_3 &\equiv \int \omega^a \wedge \left(d\omega_a + \frac{1}{3}\ \epsilon_{abc}\omega^b \wedge \omega^c\right) \\ &= \int \frac{1}{2}\ \epsilon^{abc}\omega_{bc} \wedge \left[d\left(\frac{1}{2}\ \epsilon_{ade}\omega^{de}\right) + \frac{1}{3}\ \epsilon_{abc}\left(\frac{1}{2}\ \epsilon^{bde}\omega_{de}\right) \wedge \left(\frac{1}{2}\ \epsilon^{cfg}\omega_{fg}\right)\right] \\ &= \int \frac{1}{2}\left(\omega_{bc} \wedge d\omega^{cb} + \frac{2}{3}\ \omega^b{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b\right). \end{aligned} \tag{A.9}$$

The final line is the gravitational Chern-Simons (GCS) 3-form in the tensor basis appeared in Refs. [6, 8, 10] and the first line is in the vector form basis that appeared in Refs. [66, 106, 5, 97, 65, 64], up to overall coefficients. The relation to the component (tensor) form for the spacetime indices is given by

$$\begin{aligned} I_3 &= \int \frac{1}{2}\left(\omega_{bc} \wedge R^{cb} - \frac{1}{3}\ \omega^b{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b\right) \\ &= \int \frac{1}{2}\left(\omega_{bc\mu} \cdot \frac{1}{2}R^{cb}{}_{\nu\rho} - \frac{1}{3}\ \omega^b{}_{c\mu}\omega^c{}_{d\nu}\omega^d{}_{b\rho}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= -\int d^3x\ \frac{1}{4}\ \epsilon^{\mu\nu\rho}\ \left(\omega_{bc\mu}R^{cb}{}_{\nu\rho} + \frac{2}{3}\ \omega^b{}_{c\mu}\omega^c{}_{d\nu}\omega^d{}_{b\rho}\right). \end{aligned} \tag{A.10}$$

This expression is what appeared in Refs. [1, 2, 3, 9].

Finally, I note that

$$\begin{aligned} I_4 \equiv \int \frac{1}{l^2}e^a \wedge T_a &= \int \frac{1}{l^2}e^a{}_\mu \cdot \frac{1}{2}T_{a\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= \int d^3x\ \frac{1}{l^2}\ \epsilon^{\mu\nu\rho}\ e^a{}_\mu T_{a\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho, \end{aligned} \tag{A.11}$$

where $T_{a\nu\rho} = \partial_\nu e^a{}_\rho + \epsilon^a{}_{bc}\omega^b{}_\nu e^c{}_\rho - (\nu \leftrightarrow \rho)$ is the torsion tensor. This action is what appeared in Refs. [66, 5, 65, 64].

Collecting all formulae together I arrive at the following action for the generalized CS gravity, up to the boundary term in (A.5),

$$
\begin{aligned}
I_{CS} &= \frac{k}{4\pi}\ \alpha(I_3+I_4)+\frac{k}{4\pi}\ \beta(I_1+I_2) \\
&= -\frac{k\beta}{4\pi}\int_{\mathcal{M}} d^3x\sqrt{-g}\ \left(R+\frac{2}{l^2}\right)-\frac{k\alpha}{16\pi}\int_{\mathcal{M}} d^3x\ \epsilon^{\mu\nu\rho}\ \left(\omega_{bc\mu}R^{cb}{}_{\nu\rho}+\frac{2}{3}\ \omega^b{}_{c\mu}\omega^c{}_{d\nu}\omega^d{}_{b\rho}\right) \\
&\quad +\frac{k\alpha}{8\pi}\int_{\mathcal{M}} d^3x\ \frac{1}{l^2}\ \epsilon^{\mu\nu\rho}\ e^a{}_\mu T_{a\nu\rho}. \qquad \text{(A.12)}
\end{aligned}
$$

This is the expression that appeared in Refs. [1, 2, 3, 9] but it is easy to compare with other expressions in Refs. [8, 10, 65, 64] from the above formulae. Now, in order that the first term becomes the ordinary Einstein-Hilbert action $I_{EH}=(1/16\pi G)\int_{\mathcal{M}}(R+2/l^2)$ with the cosmological constant $\Lambda=-1/l^2$ in (2.1) I choose $k\beta=-1/4G$ as in (4.17). Then the GCS term becomes, in several equivalent expressions,

$$
\begin{aligned}
I_{GCS} &\equiv \frac{k}{4\pi}\ \alpha I_3 \\
&= \frac{1}{64\pi G}\frac{\alpha}{\beta}\int_{\mathcal{M}} d^3x\ \epsilon^{\mu\nu\rho}\ \left(\omega_{bc\mu}R^{cb}{}_{\nu\rho}+\frac{2}{3}\ \omega^b{}_{c\mu}\omega^c{}_{d\nu}\omega^d{}_{b\rho}\right) \\
&= -\frac{1}{32\pi G}\frac{\alpha}{\beta}\int_{\mathcal{M}}\left(\omega_{bc}\wedge d\omega^{cb}+\frac{2}{3}\ \omega^b{}_{c\mu}\omega^c{}_{d\nu}\omega^d{}_{b\rho}\right) \\
&= -\frac{1}{16\pi G}\frac{\alpha}{\beta}\int_{\mathcal{M}}\omega^a\wedge\left(d\omega_a+\frac{1}{3}\ \epsilon_{abc}\omega^b\wedge\omega^c\right). \qquad \text{(A.13)}
\end{aligned}
$$

By comparing the first line with (2.2) and Refs. [1, 2, 3] and [9] (the published version), I find $\hat{\beta}=\alpha/l\beta=-1/\mu l=-\beta_S/l$ for the coefficient μ in Refs. [1, 2, 3] and β_S in Ref. [9], as I have claimed in (4.17); by comparing the second line with Ref. [8], I find $\hat{\beta}=\alpha/l\beta=-32\pi G\beta_{KL}/l$ for the coefficient β_{KL} in Ref. [8]; by comparing the third line with Refs. [65, 64], I find $\hat{\beta}=\alpha/l\beta=-16\pi G\alpha_3/l$. From these relations one can ensure that the central charges between the anomaly approaches of Refs. [8, 9] and the classical symmetry approaches of Refs. [65, 20] agree exactly, even in the presence of GCS term,

$$
\begin{aligned}
\hat{c}^{\pm}_{\rm tot} &= \frac{1}{\hbar}\left(1\mp 16\pi G\frac{\alpha_3}{l}\right)\frac{3l}{2G} \\
&= \frac{1}{\hbar}\left(1\mp 32\pi G\frac{\beta_{KL}}{l}\right)\frac{3l}{2G} \\
&= \frac{1}{\hbar}\left(1\mp\frac{\beta_S}{l}\right)\frac{3l}{2G} \\
&= \frac{1}{\hbar}\left(1\pm\hat{\beta}\right)\frac{3l}{2G}. \qquad \text{(A.14)}
\end{aligned}
$$

Appendix B. The Cardy formula and its higher-order corrections

In this appendix, I briefly review the physicist's derivation of the Cardy formula and its higher-order corrections for completeness of my discussions in this paper.

To this end, let me begin with the partition function of the conformal field theory on a torus, with the modular parameters $\tau, \bar{\tau}$ [42, 46]

$$Z[\tau, \bar{\tau}] = Tr e^{2\pi i \tau (\hat{L}_0 - \frac{\hat{c}}{24})} e^{-2\pi i \bar{\tau} (\hat{\bar{L}}_0 - \frac{\hat{\bar{c}}}{24})}. \tag{B.1}$$

This is invariant under the modular transformations $\tau \to (a\tau + b)/(c\tau + d)$ (similarly for $\bar{\tau}$), with the some integers a, b, c, d satisfying $ad - bc = 1$, and the Virasoro generators $\hat{L}_m, \hat{\bar{L}}_m$ are defined on the "plane" with central charges $\hat{c}, \hat{\bar{c}}$, with the algebras in the standard form,

$$\begin{aligned}
&[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{\hat{c}}{12} m(m^2-1)\delta_{m+n,0}, \\
&[\hat{\bar{L}}_m, \hat{\bar{L}}_n] = (m-n)\hat{\bar{L}}_{m+n} + \frac{\hat{\bar{c}}}{12} m(m^2-1)\delta_{m+n,0}, \\
&[\hat{L}_m, \hat{\bar{L}}_n] = 0.
\end{aligned} \tag{B.2}$$

The density of states $\rho(\hat{\Delta}, \hat{\bar{\Delta}})$ for the eigenvalues $\hat{L}_0 = \hat{\Delta}, \hat{\bar{L}}_0 = \hat{\bar{\Delta}}$ is given as a contour integral (I suppress the $\bar{\tau}$-dependence for simplicity, but the computation is similar to the τ-part)

$$\rho(\hat{\Delta}) = \int_C d\tau \; e^{-2\pi i (\hat{\Delta} - \frac{\hat{c}}{24})\tau} Z[\tau], \tag{B.3}$$

where the contour C encircles the origin in the complex $q = e^{2\pi i \tau}$ plane. The general evaluation of this integral would be impossible unless $Z[\tau]$ is known completely. But, due to the modular invariance of (B.1), one can easily compute its asymptotic formula through the steepest-descent approximation. In particular, (B.1) is invariant under $\tau \to -1/\tau$ [42] such that

$$Z[\tau] = Z[-1/\tau] = e^{-2\pi i (\hat{\Delta}_{\min} - \frac{\hat{c}}{24})\tau} \tilde{Z}[-1/\tau], \tag{B.4}$$

where $\tilde{Z}[-1/\tau] = Tr e^{-2\pi i (\hat{L}_0 - \hat{\Delta}_{\min})/\tau}$ approaches a constant value $\rho(\hat{\Delta}_{\min})$ as $\tau \to i0_+$, which defines the steepest-descent path for a "real" value of $\hat{\Delta} \geq \hat{\Delta}_{\min}$. With the help of this property, (B.3) is evaluated as, by expanding the integrand around the steepest-descent path τ_*,

$$\rho(\hat{\Delta}) = \int_C d\tau \; e^{\eta(\tau)} \tilde{Z}[-1/\tau] \tag{B.5}$$

$$\begin{aligned}
&= e^{\eta(\tau_*)} \tilde{Z}[-1/\tau^*] \times \int_C d\tau \, \exp\left\{ \frac{1}{2}\eta^{(2)}(\tau - \tau_*)^2 + \sum_{n\geq 3}^{\infty} \frac{1}{n!}\eta^{(n)}(\tau - \tau_*)^n \right\} \\
&\times \left[1 + \sum_{m \geq 1}^{\infty} \frac{1}{m!} \tilde{Z}^{-1} \tilde{Z}^{(m)} (\tau - \tau_*)^m \right].
\end{aligned} \tag{B.6}$$

Here $\eta(\tau) = -2\pi i\hat{\Delta}_{\rm eff}\tau + 2\pi i\hat{c}_{\rm eff}/(24\tau)$, which dominates $\tilde{Z}[1/\tau]$ in the region of interest, gets the maximum

$$\eta(\tau_*) \;=\; 2\pi\sqrt{\frac{\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff}}{6}} \tag{B.7}$$

with $\tau_* = i\sqrt{\hat{c}_{\rm eff}/24\hat{\Delta}_{\rm eff}}$ when

$$\frac{24\hat{\Delta}_{\rm eff}}{\hat{c}_{\rm eff}} \gg 1 \tag{B.8}$$

is satisfied. Here, $\eta^{(n)} = (d^n\eta/d\tau^n)|_{\tau=\tau_*}$, $\tilde{Z}^{(m)} = (d^n\tilde{Z}/d\tau^n)|_{\tau=\tau_*}$, and $\hat{c}_{\rm eff} = \hat{c} - 24\hat{\Delta}_{\rm min}$, $\hat{\Delta}_{\rm eff} = \hat{\Delta} - \hat{c}/24$; $\hat{\Delta}_{\rm min}$ is the minimum of $\hat{\Delta}$. Here, I am assuming "$\hat{c}_{\rm eff}, \hat{\Delta}_{\rm eff} > 0$" since, otherwise, the saddle-point approximation is not valid for *real* valued $\hat{c}_{\rm eff}$, $\hat{\Delta}_{\rm eff}$.

Then, in the limit of $\epsilon \to \infty$ with $\tau_* = i/\epsilon$, the higher-order correction terms in the bracket [ ] of (B.6) are exponentially suppressed as $e^{-2\pi\epsilon(\hat{\Delta}-\hat{\Delta}_{\rm min})}$, hence (B.6) is simplified as, up to the exponentially suppressing terms,

$$\rho(\hat{\Delta}) = e^{2\pi\sqrt{\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff}/6}} \times \int_C d\tau\, \exp\left\{\frac{1}{2}\eta^{(2)}(\tau-\tau_*)^2 + \sum_{n\geq 3}^{\infty}\frac{1}{n!}\eta^{(n)}(\tau-\tau_*)^n\right\}, \tag{B.9}$$

where I have used $\tilde{Z}[i\infty] = 1$. This is known as the Cardy formula [42]. Note that here I need

$$\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff} \gg 1 \tag{B.10}$$

in order that the approximation is reliable, i.e., $e^{\eta(\tau_*)}$ dominates in the integral of (B.5), as well as the condition (B.8) such as $\tilde{Z}[-1/\tau]$ is slowly varying near τ_*.

The integrals above could be evaluated by the steepest-descent method but the direct computation would be quite involved if one wants to go beyond the Gaussian integral. But fortunately there exits an *exact*, closed expression, due to Raedmacher [95], with the result [96, 99], up to the exponentially suppressed terms,

$$\rho(\hat{\Delta}) = e^{2\pi\sqrt{\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff}/6}} \times \left(\frac{\hat{c}_{\rm eff}}{96\hat{\Delta}_{\rm eff}^3}\right)^{1/4} I_1(2\pi\sqrt{\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff}/6}). \tag{B.11}$$

So, its corresponding entropy $S_{\rm stat} = \log\rho(\hat{\Delta})$ becomes, with $S_0 = 2\pi\sqrt{\hat{c}_{\rm eff}\hat{\Delta}_{\rm eff}/6}$,

$$\begin{aligned} S_{\rm stat} &= S_0 + \ln\left[\left(\frac{\hat{c}_{\rm eff}}{96\hat{\Delta}_{\rm eff}^3}\right)^{1/4} I_1(S_0)\right] \\ &= S_0 + \ln\left(\frac{\hat{c}_{\rm eff}}{96\hat{\Delta}_{\rm eff}^3}\right)^{1/4} - \frac{3}{8}S_0^{-1} + O((S_0)^{-2}), \end{aligned} \tag{B.12}$$

where $I_n(x)$ is the modified Bessel function of the first kind, and I have used its asymptotic series expansion for large x:

$$I_1(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[1 - \frac{3}{8} x^{-1} + O(x^{-2}) \right]. \tag{B.13}$$

Acknowledgments

I would like to thank Jacob Bekenstein, Jin-Ho Cho, Gungwon Kang, O-kab Kwon, Makoto Natsuume, Sergei Odintsov, Segrey Soloduhkin, and Ho-Ung Yee for useful correspondences. This work was supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number R11 - 2005- 021.

References

[1] S. Deser, R. Jackiw, and S. Templeton, "Topologically Massive Gauge Theories", Ann. Phys. (N.Y.), **140**, 372 (1982).

[2] S. Deser and X. Xiang, Phys. Lett. "Canonical formulations of full nonlinear topologically massive gravity", **B 263**, 39 (1991).

[3] S. Deser and B. Tekin, "Massive, topologically massive, models", Class. Quant. Grav. **19**, L 97 (2002) [hep-th/0203273].

[4] N. Kaloper, "Miens of the three-dimensional black hole", Phys. Rev. **D 48**, 2598 (1993) [hep-th/9303007].

[5] J.-H. Cho, "BTZ black-hole dressed in the gravitational Chern-Simons term", J. Korean. Phys. Soc. **44**, 1355 (2004) [hep-th/9811049].

[6] A. A. Garcia, F. W. Hehl, C. Heinicke and A. Macias, "Exact vacuum solution of a (1+2)-dimensional Poincare gauge theory: BTZ solution with torsion", Phys. Rev. **D 67**, 124016 (2003) [gr-qc/0302097].

[7] P. Kraus and F. Larsen, "Microscopic black hole entropy in theories with higher derivatives", JHEP **0509**, 034 (2005) [hep-th/0506176].

[8] P. Kraus and F. Larsen, JHEP, "Holographic gravitational anomalies", **0601**, 022 (2006) [hep-th/0508218].

[9] S. N. Solodukhin, "Holography with gravitational Chern-Simons", Phys. Rev. **D 74**, 024015 (2006) [hep-th/0509148].

[10] B. Sahoo and A. Sen, "BTZ black hole with Chern-Simons and higher derivative terms", JHEP **0607**, 008 (2006) [hep-th/0601228].

[11] S. W. Hawking, "Gravitational radiation from colliding black holes", Phys. Rev. Lett. **26**, 1344 (1971).

[12] J. D. Bekenstein, "Black holes and entropy", Phys. Rev. **D 7**, 2333 (1973).

[13] A. Strominger, "Black hole entropy from near horizon microstates", JHEP **9802**, 009 (1998) [hep-th/9712251].

[14] D. Birmingham, I. Sach, and S. Sen, "Entropy of three-dimensional black holes in string theory", Phys. Lett. **B 424**, 275 (1998) [hep-th/9801019].

[15] M.-I. Park, "Fate of three-dimensional black holes coupled to a scalar field and the Bekenstein-Hawking entropy", Phys. Lett. **B 597**, 237 (2004) [hep-th/0403089].

[16] S. Carlip and J. Gegenberg, "Gravitating topological matter in (2+1)-dimensions", Phys. Rev. **D 44**, 424 (1991).

[17] S. Carlip, J. Gegenberg, and R. B. Mann, "Black holes in three-dimensional topological gravity", Phys. Rev. **D 51**, 6854 (1995) [gr-qc/9410021].

[18] M. Banados, "Constant curvature black holes", Phys. Rev. **D 57**, 1068 (1998) [gr-qc/9703040].

[19] M. Banados, "Anti-de Sitter space and black holes", Class. Quant. Grav. **15**, 3575 (1998) [hep-th/9805087].

[20] M.-I. Park, "Thermodynamics of exotic black holes, negative temperature, and Bekenstein-Hawking entropy", hep-th/0602114.

[21] F. Larsen, "A String model of black hole microstates", Phys. Rev. **D 56**, 1005 (1997) [hep-th/9702153].

[22] K. A. Moussa, G. Clément, and C. Leygnac, "The Black holes of topologically massive gravity", Class. Quant. Grav. **20**, L 277 (2003) [gr-qc/0303042].

[23] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional spacetime", Phys. Rev. Lett. **69**, 1849 (1992) [hep-th/9204099].

[24] M. Banados, M. Henneaux, C. Teitelboim and J, Zanelli, "Geometry of the (2+1) black hole", Phys. Rev. **D 48**, 1506 (1993) [gr-qc/9302012].

[25] S. Deser, I. Kanik, and B. Tekin, Class. Quant. Grav. "Conserved charges of higher D Kerr-AdS spacetimes", **22**, 3383 (2005) [gr-qc/0506057].

[26] S. Ölmez, Ö. Sariogly and B. Tekin, "Mass and angular momentum of asymptotically ads or flat solutions in the topologically massive gravity", Class. Quant. Grav. **22**, 4355 (2005) [gr-qc/0507003].

[27] T. Jacobson, G. Kang, and R. C. Myers, "On black hole entropy", Phys. Rev. **D 49**, 6587 (1994) [gr-qc/9312023].

[28] V. Iyer and R. W. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy", Phys. Rev. **D 50**, 846 (1994) [gr-qc/9403028].

[29] T. Jacobson, G. Kang, and R. C. Myers, "Increase of black hole entropy in higher curvature gravity", Phys. Rev. **D 52**, 3518 (1995) [gr-qc/9503020].

[30] A. R. Steif, "The Quantum stress tensor in the three-dimensional black hole", Phys. Rev. **D 49**, R 585 (1994) [gr-qc/9308032].

[31] V. Balasubramanian and T. S. Levi, "Beyond the veil: Inner horizon instability and holography", Phys. Rev. **D 70**, 106005 (2004) [hep-th/0405048].

[32] see, for example, C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons Inc., New York, 1967) and references therein.

[33] R.-G. Cai, Z.-J. Lu, and Y.-Z. Zhang, "Critical behavior in (2+1)-dimensional black holes", Phys. Rev. **D 55**, 853 (1997) [gr-qc/9702032].

[34] G. W. Gibbons, M. J. Perry, and C. N. Pope, "The First law of thermodynamics for Kerr-anti-de Sitter black holes", Class. Quant. Grav. **22**, 1503 (2005) [hep-th/0408217].

[35] G. Barnich and G. Compére, "Generalized Smarr relation for Kerr AdS black holes from improved surface integrals. ", Phys. Rev. **D 71**, 044016 (2005) [gr-qc/0412029].

[36] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity", Phys. Rept. **323**, 183 (2000) [hep-th/9905111] and references therein.

[37] M. Henningson and K. Skenseris, "The Holographic Weyl anomaly", JHEP **9807**, 023 (1998) [hep-th/9806087].

[38] S. Hyun, W. T. Kim, and J. Lee, "Statistical entropy and AdS / CFT correspondence in BTZ black holes", Phys. Rev. **D 59**, 084020 (1999) [hep-th/9811005].

[39] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity", Commun. Math. Phys. **208**, 413 (1999) [hep-th/9902121].

[40] H. Terashima, "Path integral derivation of Brown-Henneaux's central charge", Phys. Rev. **D 64**, 064016 (2001) [hep-th/0102097].

[41] M. Banados, O. Chandia, and A. Ritz, "Holography and the Polyakov action", Phys. Rev. **D 65**, 126008 (2002) [hep-th/0203021].

[42] J. A. Cardy, "Operator Content Of Two-Dimensional Conformally Invariant Theories", Nucl. Phys. **B 270**, 186 (1986).

[43] S. Carlip, "Entropy from conformal field theory at Killing horizons", Class. Quant. Grav. **16**, 3327 (1999) [gr-qc/9906126].

[44] M.-I. Park, "Hamiltonian dynamics of bounded space-time and black hole entropy: Canonical method", Nucl. Phys. **B 634**, 339 (2002) [hep-th/0111224].

[45] G. Kang, J.-I. Koga, and M.-I. Park, "Near horizon conformal symmetry and black hole entropy in any dimension", Phys. Rev. **D 70**, 024005 (2004) [hep-th/0402113].

[46] M.-I. Park, "Testing holographic principle from logarithmic and higher order corrections to black hole entropy", JHEP **0412**, 041 (2004) [hep-th/0402173].

[47] J. Preskill, P. Schwarz, A. D. Shapere, S. Trivedi, and F. Wilczek, "Limitations on the statistical description of black holes", Mod. Phys. Lett. **A 6**, 2353 (1991).

[48] M. Banados, "Three-dimensional quantum geometry and black holes", hep-th/9901148.

[49] S. W. Hawking, G. T. Horowitz and S. F. Ross, "Entropy, area, and black hole pairs", Phys. Rev. **51**, 4302 (1995)[gr-qc/9409013].

[50] C. Teitelboim, "Action and entropy of extreme and nonextreme black holes", Phys. Rev. **D 51**, 4315 (1995) [hep-th/9410103].

[51] G. W. Gibbons and R. E. Kallosh, "Topology, entropy and Witten index of dilaton black holes", Phys. Rev. **D 51**, 2839 (1995)[hep-th/9407118].

[52] R. M. Wald, "'The Nernst theorem' and black hole thermodynamics", Phys. Rev. **D 56**, 6467 (1997) [gr-qc/9704008].

[53] J. D. Brown and M. Henneaux, "Central Charges In The Canonical Realization Of Asymptotic Symmetries: An Example From Three-Dimensional Gravity", Commun. Math. Phys. **104**, 207 (1986).

[54] M. Banados, T. Brotz, and M. E. Ortiz, "Boundary dynamics and the statistical mechanics of the (2+1)-dimensional black hole", Nucl. Phys. **B 545**, 340 (1999) [hep-th/9802076].

[55] P. Oh and M.-I. Park, "Symplectic reduction and symmetry algebra in boundary Chern-Simons theory", Mod. Phys. Lett. **A 14**, 231 (1998) [hep-th/9805178].

[56] M.-I. Park, "Statistical entropy of three-dimensional Kerr-de Sitter space", Phys. Lett. **B 440**, 275 (1998) [hep-th/9806119].

[57] M.-I. Park, "Symmetry algebras in Chern-Simons theories with boundary: Canonical approach", Nucl. Phys. **B 544**, 377 (1999) [hep-th/9811033].

[58] P. A. M. Dirac, *Lectures on quantum mechanics* (Yeshiva University Press, New York 1964).

[59] H. Saida and J. Soda, "Statistical entropy of BTZ black hole in higher curvature gravity", Phys. Lett. **B 471**, 358 (2000) [gr-qc/9909061].

[60] M. Natsuume, T. Okamura, and M. Sato, "Three-dimensional gravity with conformal scalar and asymptotic Virasoro algebra", Phys. Rev. **D 61**, 104005 (2000)[hep-th/9910105].

[61] M. Natsuume and T. Okamura, "Entropy for asymptotically AdS(3) black holes", Phys. Rev. **D 62**, 064027 (2000)[hep-th/9911062].

[62] M. Henneaux, C. Martínez, R. Troncoso, and J. Zanelli, "Black holes and asymptotics of 2+1 gravity coupled to a scalar field", Phy. Rev. **D 65**, 104007 (2002) [hep-th/0201170].

[63] J. Gegenberg, C. Martínez, and R. Troncoso, "A Finite action for three-dimensional gravity with a minimally coupled scalar field", Phys. Rev. **D 67**, 084007 (2003)[hep-th/0301190].

[64] M. Blagojevic and B. Cvetkovic, "Canonical structure of 3-D gravity with torsion", gr-qc/0412134.

[65] M. Blagojevic and B. Cvetkovic, "3-D gravity with torsion as a Chern-Simons gauge theory", Phys. Rev. **D 68**, 104023 (2003)[gr-qc/0307078].

[66] E. Witten, "(2+1)-Dimensional Gravity As An Exactly Soluble System", Nucl. Phys. **B 311**, 46 (1988).

[67] P. G. Bergmann and A. B. Komar, "Poisson brackets between locally defined observables in general relativity", Phys. Rev. Lett. **4**, 432 (1960).

[68] M.-I. Park and Y.-J. Park, "NonAbelian Proca model based on the improved BFT formalism", Int. J. Mod. Phys. **A 13**, 2179 (1998) [hep-th/9702134].

[69] S. Carlip, "Liouville lost, Liouville regained: Central charge in a dynamical background", Phys. Lett. **B508**, 168 (2001) [gr-qc/0103100].

[70] E. Buffenoir and K. Noui, "Unfashionable observations about three-dimensional gravity", gr-qc/0305079.

[71] O. Miskovic, "Dynamics of Wess-Zumino-Witten and Chern-Simons theories", hep-th/0401185.

[72] F. Aralan, H. Arfaei, and M. M. Sheikh-Jabbari, "Dirac quantization of open strings and noncommutativity in branes", Nucl. Phys. **B 576**, 578 (2000) [hep-th/9906161].

[73] C.-S. Chu and P.-M. Ho, "Constrained quantization of open string in background B field and noncommutative D-brane", Nucl. Phys. **B 568**, 447 (2000) [hep-th/9906192].

[74] W. T. Kim and J. J. Oh, "Noncommutative open strings from Dirac quantization", Mod. Phys. Lett. **A 15**, 1597 (2000) [hep-th/9911085].

[75] T. Lee, "Canonical quantization of open string and noncommutative geometry", Phys. Rev. **D 62**, 024022 (2000) [hep-th/9911140].

[76] M. Zabzine, "Hamiltonian systems with boundaries", JHEP **0010**, 042 (2000) [hep-th/0005142].

[77] M. Bañados, "Global charges in Chern-Simons field theory and the (2+1) black hole", Phys. Rev. **D 52**, 5816 (1996) [hep-th/9405171].

[78] T. Regge and C. Teitelboim, "Role Of Surface Integrals In The Hamiltonian Formulation Of General Relativity", Ann. Phys. (N.Y.) **88** , 286 (1974).

[79] R. Benguria, P. Cordero, and C. Teitelboim, “Aspects Of The Hamiltonian Dynamics Of Interacting Gravitational Gauge And Higgs Fields With Applications To Spherical Symmetry”, Nucl. Phys. **B 122**, 61 (1977).

[80] J. D. Brown and M. Henneaux, “On The Poisson Brackets Of Differentiable Generators In Classical Field Theory”, J. Math. Phys. **27**, 489 (1986).

[81] S. Silva, “On superpotentials and charge algebras of gauge theories”, Nucl. Phys. **B 558**, 391 (1999) [hep-th/9809109].

[82] G. Barnich and F. Brandt, “Covariant theory of asymptotic symmetries, conservation laws and central charges”, Nucl. Phys. **B 633**, 3 (2002) [hep-th/0111246].

[83] M.-I. Park and Y.-J. Park, “New gauge invariant formulation of the Chern-Simons gauge theory”, Phys. Rev. **D 58**, 101702 (1998) [hep-th/9803208].

[84] G. 't Hooft, “The black hole interpretation of string theory”, Nucl. Phys. **B 335**, 138 (1990).

[85] L. Susskind, “Some speculations about black hole entropy in string theory”, hep-th/9309145.

[86] L. Susskind and J. Uglum, “Black hole entropy in canonical quantum gravity and superstring theory”, Phys. Rev. **D 50**, 2700 (1994) [hep-th/9401070].

[87] A. Sen, “Extremal black holes and elementary string states”, Mod. Phys. Lett. **A 10** 2081 (1995) [hep-th/9504147].

[88] S. Carlip, “What we don't know about BTZ black hole entropy”, Class. Quant. Grav. **15**, 3609 (1998) [hep-th/9806026].

[89] P. D. Francesco et. al., *Conformal Field Theory* (Springer-Verlag, New York, 1997).

[90] H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Jr., “Charged black hole in a grand canonical ensemble”, Phys. Rev. **D 42**, 3376 (1990).

[91] For a non-technical review, see T. Mohaupt, “Strings, higher curvature corrections, and black holes”, hep-th/0512048.

[92] G. 't Hooft and M. Veltman, “One loop divergencies in the theory of gravitation”, Ann. Poincare Phys. Theor. **A 20**, 69 (1974).

[93] M. Goroff and A. Sagnotti, "The Ultraviolet Behavior Of Einstein Gravity", Nucl. Phys. **B 266**, 709 (1986).

[94] A. E. M. van de Ven, "Two loop quantum gravity", Nucl. Phys. **B 378**, 309 (1992).

[95] H. Rademacher, "The Fourier coefficients of the modular invariant $J(\tau)$ ", Amer. J. Math. **60**, 501 (1938).

[96] R. Dijkgraaf, J. Maldacena , G. Moore, and E. Velinde, "A Black hole Farey tail", hep-th/0005003.

[97] E. W. Mielke and A. A. R. Maggiolo, "Rotating black hole solution in a generalized topological 3-D gravity with torsion", Phys. Rev. **D 68**, 104026 (2003).

[98] S. Carlip, "The Statistical mechanics of the three-dimensional Euclidean black hole", Phys. Rev. **D55**, 878 (1997) [gr-qc/9606043].

[99] D. Birmingham and S. Sen, "An Exact black hole entropy bound", Phys. Rev. **D 63**, 047501 (2001) [hep-th/0008051].

[100] M. Blagojevic and B. Cvetkovic, "Black hole entropy in 3-D gravity with torsion", Class. Quant. Grav. **23**, 4781 (2006) [gr-qc/0601006].

[101] M. Blagojevic and B. Cvetkovic, "Covariant description of the black hole entropy in 3D gravity", gr-qc/0607026.

[102] K. Maeda, M. Natsuume, and T. Okamura, "Extracting information behind the veil of horizon", hep-th/0605224.

[103] R. C. Myers and M. J. Perry, "Black Holes In Higher Dimensional Space-Times", Ann. Phys. (N. Y.) **172**, 304 (1986).

[104] R. Emparan, R. C. Myers, " Instability of ultra-spinning black holes", JHEP **0309**, 025 (2003) [hep-th/0308056].

[105] H. K. Kunduri, J. Lucietti, and H. S. Reall, "Gravitational perturbations of higher dimensional rotating black holes", hep-th/0606076.

[106] J. H. Horne and E. Witten, "Conformal Gravity In Three-Dimensions As A Gauge Theory", Phys. Rev. Lett. **62**, 501 (1989).

[107] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago and London, 1984).

[108] M. Banados and F. Mendez, “A Note on covariant action integrals in three-dimensions”, Phys. Rev. **D 58**, 104014 (1998) [hep-th/9806065].

[109] P. Miskovic and R. Olea, “On boundary conditions in three-dimensional AdS gravity”, hep-th/0603092.